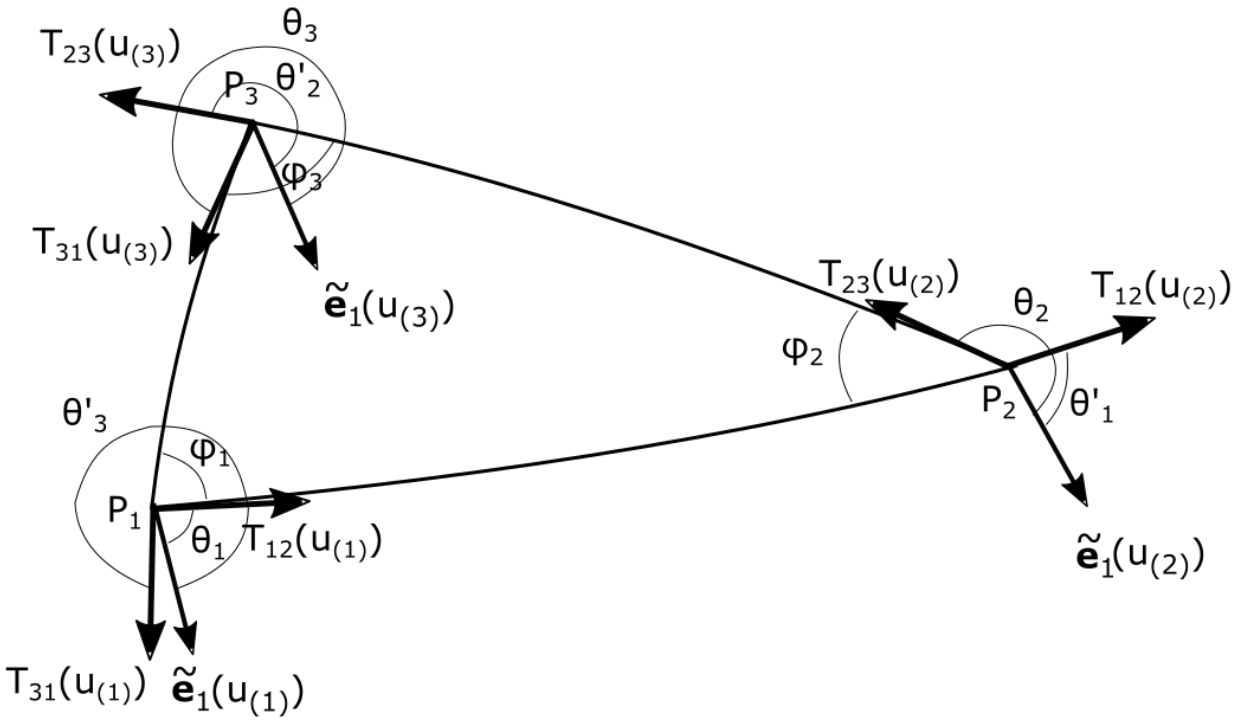
An Elementary Introduction to the Riemannian Geometry of Surfaces for Physicists

K. G. Papamichalis



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To Katerina, Vassiliki,
George, Konstantinos

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Preface

The subject of this work is the study and the comprehension of the basic properties of a Riemannian surface [(1), (2), (3)], by using almost elementary mathematical concepts. The goal of the author is to offer to the reader a path to understanding the basic principles of the Riemannian Geometry that reflects his own path to this objective. In principle, this book is addressed mainly to physicists or engineers with a good background on Mathematical Analysis and Linear Algebra, as well as to any student interested in Differential Geometry and its applications. Although the present work has the characteristics of a "standard text", certainly it is not a strictly mathematical essay; the principal concern of the author is to put a sequence of milestones leading to the heart of a Riemannian geometric structure. As a consequence, the proofs of the emerging theorems are outlined in the form of "steps-to-the-proof": a sequence of steps is displayed that focus on the main concepts and ideas leading to the final result; but certainly they do not have the status of a strict mathematical proof.
The adopted route of inquiry presupposes a certain knowledge and adeptness on some mathematical concepts and technics at a level determined by the relative references [(4)-(8)]. The language and the mathematical context which is necessary for the description of the properties of the Riemannian surfaces are gradually been building starting from the presupposed knowledge. The geometric features of an abstract geometric surface are developed as a generalization or prolongation of the corresponding features of a surface immersed in a three dimensional Euclidean or pseudo-Euclidean space [(1), (2), (3)].

The whole work has been divided in 15 "paragraphs", each subdivided in a number of "sections". A number of "examples" has been included in most of the paragraphs, aiming at the application of the most important results to geometric structures familiar from the elementary geometry. The set of the paragraphs has been separated in two "chapters".
In chapter 1 we concern with the fundamental concepts identifying a 3-dimensional Euclidean or Minkowski space and we focus to the key-ideas that will help us to build the structure of an abstract geometric surface. The "building blocks" we borrow from the structure of the Euclidean space, to achieve our goal are the following: "the tangent spaces of a Euclidean space", "the metric tensor on the tangent spaces" "1, 2 and 3-forms", "coordinate transformations", "invariant forms".
We start with the study of the geometry of a 3-dimensional Euclidean space and incidentally of the 3-dimensional Minkowski space. We introduce the concept of the tangent spaces and the inner product in them, induced by the Euclidean inner product of the underlying space. Then we define the concept of the "forms" on the tangent spaces and examine how they transform under a coordinate-transformation of the underlying space. We focus on the properties and the construction of the group of isometric coordinate-transformations on the three-dimensional Euclidean or pseudo-Euclidean space. The concepts "area" and "volume" are defined as invariant forms under the group of the isometric coordinate-transformations.

In chapter 2 we concern with the definition, the properties and the features of a surface. We begin with the idea of a surface immersed in a 3-dimensional Euclidean or pseudo-Euclidean space (paragraphs 7, 8, 9). We try to describe the geometry of these surfaces by applying all the ideas, the reasoning paths and the geometric concepts we obtained in the previous chapter studding the Euclidean spaces, having the ultimate goal to get the surface rid from its underlying space. In the examples we manage in paragraph 9, we apply the induced relations and propositions for the case of a surface of revolution. In paragraph 10 we make the crucial abstraction to the idea of the Riemannian or geometric surface which is not necessarily

immersed in any underlying space. Then we gradually define the fundamental concepts and relations which determine the structure of any geometric surface: “tangent planes”, “inner product”, “connection”, “covariant differentiation”, “parallel displacement”, “curvature”, “geodesic curves”, "frame fields", "connection forms", "geodesic curvature" and “geodesic polygons”.
The examples aim at applying the general relations we have obtained, to the special cases of geometric surfaces with structure similar to surfaces of revolution and especially to a sphere.
In the Appendices we develop a procedure aiming to construct the groups of the coordinate-transformations which leave invariant a given real function defined on the tangent spaces of a Euclidean or pseudo-Euclidean space.

Konstantinos G. Papamichalis

Symbolism

Everywhere in the mathematical expressions, we follow the Einstein convention for the summation of quantities depended on indices [5].

$\boldsymbol{R}$: The set of the real numbers

$\boldsymbol{R}_0^3$: The 3-dimensional Euclidean space. It consists of the triples:

$x = \left(x^1, x^2, x^3\right)$, $x^j \in \boldsymbol{R}$, $j = 1,2,3$

It has the structure of a linear space equipped with the inner product:

$x \cdot y = x^1 y^1 + x^2 y^2 + x^3 y^3$

$\boldsymbol{R}_1^3$: The 3-dimensional pseudo-Euclidean space. It consists of the triples:

$x = \left(x^1, x^2, x^3\right)$, $x^j \in \boldsymbol{R}$, $j = 1,2,3$

It has the structure of a linear space equipped with the inner product:

$x \cdot y = x^1 y^1 + x^2 y^2 - x^3 y^3$

In general, the symbol $\boldsymbol{R}_m^n$ (n, m integers: $n > m$) suggests an n-dimensional pseudo-Euclidean space equipped with an inner product of the form:

$x \cdot y = x^1 y^1 + x^2 y^2 + \ldots + x^{n-m} y^{n-m} - x^{n-m+1} y^{n-m+1} - \ldots - x^n y^n$

$L\left(\left\{\xi_{(1)}, \xi_{(2)}\right\}\right)$, $\xi_{(1)}, \xi_{(2)} \in V$: The subspace of the linear space V spanned by the linear combinations:

$\xi_{(1)} a + \xi_{(2)} \beta \in L\left(\left\{\xi_{(1)}, \xi_{(2)}\right\}\right) \subseteq V$, $a, \beta \in \boldsymbol{R}$

$T_x \boldsymbol{R}_0^3$, $T_P \boldsymbol{R}_0^3$: The tangent space of $\boldsymbol{R}_0^3$ at its point P determined by the tuple $x = \left(x^1, x^2, x^3\right)$

$$\dot{x}(t) = \left(\dot{x}^1(\mathrm{t}), \dot{x}^2(\mathrm{t}), \dot{x}^3(t)\right) = \frac{dx(t)}{dt} = \left(\frac{dx^1(t)}{dt}, \frac{dx^2(t)}{dt}, \frac{dx^3(t)}{dt}\right)$$

$\Delta x = \dot{x}(t) \Delta t$, $\Delta t \to 0$: An infinitesimal vector of the tangent space $T_x \boldsymbol{R}_0^3$, $x = x(t)$

$\boldsymbol{x}_j$, $j = 1,2,3$: The "natural" basis vectors defining a Cartesian coordinate system of the Euclidean space $\boldsymbol{R}_0^3$ and its tangent spaces $T_x \boldsymbol{R}_0^3$ ($x \in \boldsymbol{R}_0^3$). They are determined by the relations:

$\boldsymbol{x}_1 = (1,0,0)$, $\boldsymbol{x}_2 = (0,1,0)$, $\boldsymbol{x}_3 = (0,0,1)$

$\boldsymbol{x}_j \cdot \boldsymbol{x}_k = \delta_{jk}$

dx_j, $j = 1,2,3$: The basic 1-forms on the tangent spaces of $\boldsymbol{R}_0^3$ defined by the relationship:

$dx_j(\Delta x) \underset{def}{=} \boldsymbol{x}_j \cdot \Delta x$

dx^j, $j = 1,2,3$: The basic 1-forms on the tangent spaces of $\boldsymbol{R}_0^3$ defined by the relationship:

$dx^j(\Delta x) = \Delta x^j$

ω^{μ}, $\mu = 1,2$: The basic 1-forms on the tangent spaces of a geometric surface *S* defined by the relationship:
$\omega^{\mu}\left(e_{v}(u)\xi^{v}\right) = \xi^{\mu}$

ω_{μ}, $\mu = 1,2$: The basic 1-forms on the tangent spaces of a geometric surface *S* defined by the relationship:
$\omega_{v}\left(e_{\mu}(u)\xi^{\mu}\right) = \left\langle e_{v}(u), e_{\mu}(u)\right\rangle \xi^{\mu} = g_{v\mu}(u)\xi^{\mu}$

$\bar{\Gamma}_{v\kappa\lambda}$, $\bar{\Gamma}^{v}_{\kappa\lambda}$, $\Gamma_{v\kappa\lambda}$, $\Gamma^{v}_{\kappa\lambda}$ $v,\kappa,\lambda = 1,2$: The Christoffel symbols, determined by a connection defined on a geometric surface *S*

$D_{\Delta u}\bar{\xi}(u) = \bar{\varphi}_{u,u+\Delta u}\left(\bar{\xi}(u+\Delta u)\right) - \bar{\xi}(u) = e_{\mu}(u) D^{\mu}_{\Delta u}\bar{\xi}(u)$, $D^{\mu}_{\Delta u}\bar{\xi}(u) = \left(\frac{\partial\bar{\xi}^{\mu}(u)}{\partial u^{\kappa}} + \bar{\Gamma}^{\mu}_{v\kappa}(u)\bar{\xi}^{v}(u)\right)\Delta u^{\kappa}$: The covariant differential of a vector field $\bar{\xi}(u)$ with respect to the connection $\bar{\varphi}$ defined on a geometric surface *S*

$R_{v\kappa} = -\partial_1\bar{\Gamma}_{v\kappa 2} + \partial_2\bar{\Gamma}_{v\kappa 1} - \bar{\Gamma}^{\lambda}_{\kappa 1}\bar{\Gamma}_{\lambda v 2} + \bar{\Gamma}^{\lambda}_{\kappa 2}\bar{\Gamma}_{\lambda v 1}$: The curvature matrix of a geometric surface S

$K(P) = \frac{1}{\det g}\left(-\partial_1\bar{\Gamma}_{212} + \partial_2\bar{\Gamma}_{211} + \bar{\Gamma}_{\lambda 12}\bar{\Gamma}^{\lambda}_{12} - \bar{\Gamma}_{\lambda 22}\bar{\Gamma}^{\lambda}_{11}\right)$: The (Gaussian) curvature, determined at any point *P* of a geometric surface *S*

$\tilde{\omega}_{\mu v}(\Delta U) \underset{def}{=} \left\langle \tilde{e}_{\mu}, D_{\Delta u}\tilde{e}_{v}\right\rangle$, $\Delta U = \tilde{e}_{\mu}(u)\Delta u^{\mu}$: The connection forms on a geometric surface *S*. The basis-elements $\tilde{e}_{\mu}(u)$, $\mu = 1,2$ constitute a frame field on the tangent spaces of *S*

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Chapter 1

Key concepts

3-dimensional Euclidean space - Inner product - Tangent spaces - 1, 2 and 3-forms - Integration of the 1 and 2-forms - Exterior derivatives of the 1 and 2-forms - Coordinate transformations - Vector fields - Invariant functions and forms on the tangent spaces of a Euclidean or pseudo-Euclidean space - The area-form - The volume-form

1. **The 3-dimensional Euclidean and Minkowski space**

In this, first paragraph the main features of a 3-dimensional Euclidean or pseudo-Euclidean space are described: a) the determination of its points in Cartesian and non-Cartesian coordinates, b) the operations of the addition and scalar multiplication with any real number, c) the definition of the "natural" basis, d) the Euclidean inner product and the corresponding metric tensor in the Euclidean and the Minkowski space e) the definition of the norm and the distance in the Euclidean and the Minkowski space.

The 3-dimensional Euclidean space (symbolized: $\boldsymbol{R}_0^3$) is defined as the set of the triples [2] $x=(x^1,x^2,x^3)$, $x^j \in \boldsymbol{R}$, $j=1,2,3$ equipped with a structure characterized by the following features:

A) In $\boldsymbol{R}_0^3$ the operations of addition and scalar multiplication by a real number have been defined:

For any $x=\left(x^1,x^2,x^3\right)\in \boldsymbol{R}_0^3$, $y=\left(y^1,y^2,y^3\right)\in \boldsymbol{R}_0^3$, $\lambda \in \boldsymbol{R}$ we define:

$$x+y=\left(x^1+y^1,x^2+y^2,x^3+y^3\right)\in \boldsymbol{R}_0^3$$

$$\lambda x=\left(\lambda x^1,\lambda x^2,\lambda x^3\right)\in \boldsymbol{R}_0^3$$

We can easily verify that $\boldsymbol{R}_0^3$ equipped with the previous operations, has the structure of a linear (vector) space [2], [5]. The numbers x^1, x^2, x^3 determining each point x of the space are called "**Cartesian coordinates** of x". The point O determined by the triple $(0,0,0)$ is called the **origin** of the Euclidean space.

Remark: The above definition of the addition and scalar multiplication for the triples $x=(x^1,x^2,x^3)$ specify the coordinates x^1, x^2, x^3 as "Cartesian". We could for example, imagine another coordinate system (let us call it "polar coordinate system") such that each point x of $\boldsymbol{R}_0^3$ is determined by another triple, say (ρ,φ,ζ) where the coordinates ρ,φ,ζ are related with the Cartesians x^1, x^2, x^3 by the transformation:

$x^1=\rho\cos\varphi$, $x^2=\rho\sin\varphi$, $x^3=\zeta$

In the polar coordinate system, let the points x and y be determined by the triples (ρ,φ,ζ) and (ρ',φ',ζ') respectively; their addition $x+y$ is determined by the triple $(\rho'',\varphi'',\zeta'')$ where:

$$\rho''=\sqrt{\rho^2+\rho'^2+2\rho\rho'\cos(\varphi-\varphi')}$$

$$\varphi''=\tan^{-1}\left(\frac{\rho\sin\varphi+\rho'\sin\varphi'}{\rho\cos\varphi+\rho'\cos\varphi'}\right)$$

$$\zeta''=\zeta+\zeta'$$

B) The Euclidean space $\boldsymbol{R}_0^3$ is equipped with the Euclidean inner product (or dot product), defined by the bilinear form:

$\boldsymbol{R}_0^3 \otimes \boldsymbol{R}_0^3 \ni (x, y) \to x \cdot y \in \boldsymbol{R}$

In Cartesian coordinates the analytic expression of the Euclidean inner product is given by the relation:

$x \cdot y = x^1 y^1 + x^2 y^2 + x^3 y^3$

Remark: Notice that in the polar coordinate system, the Euclidean inner product is expressed by the relation:

$x \cdot y = \rho\rho' \cos(\varphi - \varphi') + \zeta\zeta'$

In the Cartesian coordinate system we define the "natural basis" of the 3-dimensional Euclidean space to be the basis-triples:

$\boldsymbol{x}_1 = (1,0,0)$, $\boldsymbol{x}_2 = (0,1,0)$, $\boldsymbol{x}_3 = (0,0,1)$

Any element $x = (x^1, x^2, x^3) \in \boldsymbol{R}_0^3$ is expressed as a linear combination of the basis-triples:

$x = \boldsymbol{x}_j x^j$, $j = 1,2,3$

We can easily verify that:

$\boldsymbol{x}_j \cdot \boldsymbol{x}_k = \delta_{jk}$

The symbol δ_{jk} is defined as the Kronecker's delta[7].

The linearity of the inner product implies that:

$x \cdot y = (x^j \hat{x}_j) \cdot (y^k \hat{x}_k) = x^j y^k \hat{x}_j \cdot \hat{x}_k = \delta_{jk} x^j y^k$

In general the inner products of the basis-elements of an inner-product-space *V*, determine a matrix with elements:

$g_{jk} = \boldsymbol{x}_j \cdot \boldsymbol{x}_k$

The matrix $[g_{jk}]$ is called the "**metric tensor**" of *V*. Hence, the metric tensor of the Euclidean space $\boldsymbol{R}_0^3$ in Cartesian coordinates is the identity matrix *I*:

$[g_{jk}] = [\delta_{jk}] \underset{def}{=} I$

What about the metric tensor of the 3-dimensional Minkowski space?

In the Minkowski space $\boldsymbol{R}_1^3$ the matrix-elements of the metric tensor in Cartesian coordinates, are determined by the relationships:

$\boldsymbol{x}_j \cdot \boldsymbol{x}_k = 0$ if $j \neq k$

$\boldsymbol{x}_1 \cdot \boldsymbol{x}_1 = 1$, $\boldsymbol{x}_2 \cdot \boldsymbol{x}_2 = 1$, $\boldsymbol{x}_3 \cdot \boldsymbol{x}_3 = -1$

Hence, the metric tensor of a Minkowski space in Cartesian coordinates is determined by the matrix:

$$g \underset{def}{=} [g_{jk}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

In any case, **the metric tensor is symmetric**. By definition, in Riemann spaces the inner product is positive defined [3], but in Minkowski or Einstein spaces is not.

The inverse matrix of $[g_{jk}]$ is symbolized by $[g^{jk}]$

It holds:

$$\left[g^{jk}\right]\cdot\left[g_{kn}\right]=I$$
$$g^{jk}g_{kn}=\delta_n^j$$

We define the **norm** of a point *x* in an inner-product-space, by the relationship:
$|x|\underset{def}{=}\sqrt{x\cdot x}=\sqrt{g_{jk}x^jx^k}$
The **distance *d*(x,*y*)** between two points *x, y* of a space with norm, is defined as follows:
$d(x,y)\underset{def}{=}|x-y|=\sqrt{(x-y)\cdot(x-y)}$
We notice that the measurement of distances in a space with norm, as well as the calculation of distances between points on curves or surfaces immersed in this space, is possible only if we know the metric tensor *g*(*x*) of the underlying space.

Remark: We could get an abstraction of the idea of the Euclidean space, as follows: Consider a set *V* of points (whatever these points could be) such that there is a one-one correspondence of *V* on the triples of the Euclidean space $\boldsymbol{R}_0^3$ [(3)]. Then the structure of $\boldsymbol{R}_0^3$ is projected on *V* and we could claim that *V* is a three-dimensional Euclidean space [(3)]. Assume that the points of $\boldsymbol{R}_0^3$ are determined in Cartesian coordinates. Name *O* the point of *V* corresponding to the triple (0,0,0); we call *O* "the origin" of *V*. Then, name P_1, P_2, P_3 the points of *V* corresponding to the triples:
$\boldsymbol{x}_1=(1,0,0)$, $\boldsymbol{x}_2=(0,1,0)$, $\boldsymbol{x}_3=(0,0,1)\in\boldsymbol{R}_0^3$
The origin *O* and the points P_1, P_2, P_3 of *V* determine a "**Cartesian coordinate system**" on *V*.

2. **Tangent spaces**
In the second paragraph, we introduce the idea of the tangent spaces of a Euclidean or pseudo-Euclidean space. This concept, because of the "flat" structure of a Euclidean space, at first glance seems to be unnecessary, trivial or superfluous. Nevertheless, its generalization to the non-Euclidean cases and to surfaces proves to be of crucial importance for the development of a mathematical arsenal suitable to describe the features of these structures.

The idea of the tangent spaces of $\boldsymbol{R}_0^3$ comes from the notion of the tangent vectors of a curve. A curve *c* of the space $\boldsymbol{R}_0^3$ is a subset of $\boldsymbol{R}_0^3$ whose points are determined by a differentiable function of the form:
$c:I\ni t\to x_t\underset{def}{=}x(t)=\left(x^1(t),x^2(t),x^3(t)\right)\in\boldsymbol{R}_0^3$
The domain of the parameter *t* is an interval *I* of the real numbers $\boldsymbol{R}$

The curve *c* is well-defined under two assumptions [(1), (2), (3)]:
(a) We accept that the constant function, which maps the whole *I* at one fixed point of $\boldsymbol{R}_0^3$ is a -degenerate- curve; we call it "the constant curve".
(b) For any curve different of the constant curve, we assume that for any $t\in I$ there is an index $j\in\{1,2,3\}$ for which it holds:

$$\dot{x}^j(t)=\frac{dx^j(t)}{dt}\neq 0$$

I.e. the tangent vector of a curve is nowhere identically zero.

The tangent vector of the curve c at its point $P \equiv P_{x(t)}$ determined by the triple $x=x(t)$ is defined by the derivative:

$$\dot{x}_t = \dot{x}(t) \underset{def}{=} \frac{d}{dt}\left(x^1(t), x^2(t), x^3(t)\right) = \left(\dot{x}^1(t), \dot{x}^2(t), \dot{x}^3(t)\right)$$

We make the parameter transformation: $t = \lambda\tau$, $\lambda = \text{constant} \in \boldsymbol{R}$

The new parameter τ runs the interval: $\tilde{I} \subset \boldsymbol{R}$

The abstract point P of the curve c is represented by the identical triples $\tilde{x}(\tau)$ and $x(t)$:

$$\tilde{x}(\tau) = x(t) = x(\lambda\tau)$$

The tangent vector of the curve c at P, as a function of the parameter τ is calculated by the derivative:

$$\frac{d\tilde{x}(\tau)}{d\tau} = \frac{dx(t)}{d\tau} = \frac{dx(t)}{dt}\frac{dt}{d\tau} = \dot{x}(t)\lambda$$

We imply that every vector $\xi = \dot{x}(t)\lambda$, $\lambda \in \boldsymbol{R}$ is tangent of the curve c at its point $P_{x(t)}$. Or else: every element of the one-dimensional vector space $L\left(\{\dot{x}(t)\}\right)$ generated by the vector $\dot{x}(t)$ (see: "Symbolism") is a tangent vector of the curve c.

Let us now consider a point $P \equiv P_x \in \boldsymbol{R}_0^3$ determined by the triple x and assume **all** the curves of $\boldsymbol{R}_0^3$ that pass by P. We define the set $T_x\boldsymbol{R}_0^3$ of all the vectors $\xi(x)$ which are tangent to some curve of $\boldsymbol{R}_0^3$ passing by P:

$$\xi(x) \in T_x\boldsymbol{R}_0^3 \underset{def}{\Leftrightarrow} \text{"there is some curve } c_x : x_t = x(t) \text{ in } \boldsymbol{R}_0^3 \text{ with : } x(0) = x,\ \xi(x) = \dot{x}(0)\text{"}$$

Or:

$$T_x\boldsymbol{R}_0^3 \underset{def}{=} \left\{\xi(x) = \left(\xi^1(x), \xi^2(x), \xi^3(x)\right) : \xi(x) \text{ is the tangent vector at } x, \text{ of a curve in } \boldsymbol{R}_0^3 \text{ passing by } x\right\}$$

We symbolize $\xi^j(x)$ $j = 1,2,3$ the Cartesian coordinates of the vector $\xi(x)$ that is tangent at the point x, to a curve of $\boldsymbol{R}_0^3$ passing by x.

Proposition 2.1

The set $T_x\boldsymbol{R}_0^3$ is a vector space isomorphic to the 3-dimensional Euclidean space.

It is called "**the tangent space of $\boldsymbol{R}_0^3$ at the point x of the space**".

Steps to the proof

a) Any tangent space $T_x\boldsymbol{R}_0^3$ contains the zero-vector:

The tangent vector of the -degenerate- constant curve $c^j(t) = x^j$ is the zero triple (0,0,0).

b) Let: $\xi(x) = \boldsymbol{x}_j \cdot \xi^j(x) \in T_x\boldsymbol{R}_0^3$

Then, according to the definition of the tangent spaces, there is a curve $x_t^j = x^j(t)$ such that: $x^j(0) = x^j$, $\dot{x}^j(0) = \xi^j(x)$

We shall show that for any $\lambda \in \boldsymbol{R}$ it holds: $\xi(x)\lambda \in T_x\boldsymbol{R}_0^3$

We have:

For $\lambda = 0$ it holds: $\xi(x)\lambda = 0$

Then, according to (a): $\xi(x)\lambda \in T_x\boldsymbol{R}_0^3$

For $\lambda \neq 0$ define the curve: $\tilde{x}_t^j = \tilde{x}^j(t) = x^j(\lambda t)$

This curve passes from x:

$$\tilde{x}^j(0) = x^j(0) = x^j$$

Hence, by definition, its tangent vectors at $t = 0$ belongs to the tangent space.
We conclude that:

$$T_x\boldsymbol{R}_0^3 \ni \left.\frac{d\tilde{x}(t)}{dt}\right|_{t=0} = \left.\frac{dx(\lambda t)}{dt}\right|_{t=0} = \left.\frac{dx(\tau)}{d\tau}\right|_{\tau=0} \lambda = \xi(x)\lambda$$

c) Let:

$$\xi_{(1)}(x) = \boldsymbol{x}_j \cdot \xi^j_{(1)}(x),\ \xi_{(2)}(x) = \boldsymbol{x}_j \cdot \xi^j_{(2)}(x) \in T_x\boldsymbol{R}_0^3$$

Then, there are curves $x^j_{(1)} = x^j_{(1)}(t),\ x^j_{(2)} = x^j_{(2)}(t)$ passing from x (for t=0) with tangent vectors at x the vectors $\xi_{(1)}(x),\ \xi_{(2)}(x)$ respectively.

The curve $x^j_{(1+2)}(t) = \frac{1}{2}\left(x^j_{(1)}(2t) + x^j_{(2)}(2t)\right)$ passes from x (for t=0) and its tangent vector at x is:

$$\left.\frac{dx_{(1+2)}(t)}{dt}\right|_{t=0} = \xi_{(1)}(x) + \xi_{(2)}(x)$$

Hence:

$$\xi_{(1)}(x) + \xi_{(2)}(x) \in T_x\boldsymbol{R}_0^3$$

In the tangent space $T_x\boldsymbol{R}_0^3$ we define an inner product induced by the Euclidean inner product of the underlying 3-dimensional Euclidean space:

$$\xi_{(1)}(x) \cdot \xi_{(2)}(x) = \left(\boldsymbol{x}_j \xi^j_{(1)}(x)\right) \cdot \left(\boldsymbol{x}_k \xi^k_{(2)}(x)\right) = \boldsymbol{x}_j \cdot \boldsymbol{x}_k\, \xi^j_{(1)}(x) \cdot \xi^k_{(2)}(x) = g_{jk}(x)\xi^j_{(1)}(x) \cdot \xi^k_{(2)}(x)$$

In Cartesian coordinates the matrix $g(x) = \left[g_{jk}(x)\right]$ of the metric tensor, for any tangent space, equals the identity matrix $I = \left[\delta_{jk}\right]$ (see paragraph 1).

•

The "**length**" (or **norm**) of a vector $\xi(x) \in T_x\boldsymbol{R}_0^3$ is defined by the relationship:

$$\left|\xi(x)\right| \underset{def}{=} \sqrt{\xi(x) \cdot \xi(x)} = \sqrt{g_{jk}(x)\xi^j(x)\xi^k(x)}$$

For any $x \in \boldsymbol{R}_0^3$ the tangent space $T_x\boldsymbol{R}_0^3$ equipped with the above metric, is a **metric space**.
By the same way we are able to define the tangent spaces of a 3-dimensional Minkowski space.

Vector fields on the Euclidean or pseudo-Euclidean spaces

We define as **vector field** on a subset C of a 3-dimensional Euclidean space any vector function determined on C which for every $x \in C \subseteq \boldsymbol{R}_0^3$ returns a vector $\xi(x)$ of the tangent space at x:

$$\boldsymbol{R}_0^3 \supseteq C \ni x \to \xi(x) \in T_x\boldsymbol{R}_0^3$$

For example, the tangent vectors of a curve $c : x^j_t = c^j(t),\ t \in I \subseteq \boldsymbol{R}$ define a vector field on the subset $c(I)$ of the Euclidean space. The analytic expression of this vector field in Cartesian coordinates is determined by the relationship:

$$\xi\left(x_t\right) = \boldsymbol{x}_j \frac{dc^j(t)}{dt} \in T_{c(t)}\boldsymbol{R}_0^3$$

Remark: From the point of view of the Euclidean spaces the concept of the **vector field** seems identical to the concept of a vector function defined at the points of the space and with values in its tangent spaces. Given that the "natural" basis of the space is the same for all the tangent spaces of a Euclidean space, the variation of a vector function is determined

completely by the analytic expressions of its Cartesian coordinates. But in the case of non-Euclidean spaces the game is more complicated: the variation of the vector function depends both on the analytic expression of its coordinates and of the variation of basis-elements of the corresponding tangent space; we need to know how this basis changes when we move from one tangent space to another.
The concept of the vector field could be considered as a generalization of the vector function; it is endowed with the necessary characteristics that involve information about the structure of the underlying space and permit us to investigate its geometric features.

3. **1-forms**

The forms are linear functions defined on the tangent spaces of a Euclidean or non-Euclidean space or even a surface. The analytic expression of a form depends on the position of the tangent space of the underlying geometric structure. To investigate the way one form varies when we are moving from one tangent space to a neighboring one, we study the action of the form on vector fields defined on curves of the underlying geometric structure; from the processing of the results, significant geometric properties of the studied space or the surface emerge.
By scrutinizing on the behavior of the forms under coordinate-transformations, we find out the analytic expressions of those forms which are invariant under the group of the isometric transformations; these forms lead us to define important geometric quantities of the underlying structure like the area-element and the volume-element.

We call 1-form on $\boldsymbol{R}_0^3$ any linear function with domain the tangent spaces $T_x\boldsymbol{R}_0^3$, $x \in \boldsymbol{R}_0^3$ of $\boldsymbol{R}_0^3$ and range in $\boldsymbol{R}$.
Let ω_p be a 1-form and $\xi(x) = \boldsymbol{x}_j \cdot \xi^j(x) \in T_x\boldsymbol{R}_0^3$ a vector field defined on $\boldsymbol{R}_0^3$ (see paragraph 2).
We can always write:

$$\omega_p\left(\xi(x)\right) = p_j(x)\xi^j(x) \tag{3.1a}$$

The real functions $p_j(x)$, j=1, 2, 3 determine the analytic expression of the form.
For each form ω_p we can always define the vector field:

$$\xi_{(p)}(x) = \boldsymbol{x}_j\, \xi^j_{(p)}(x),\ \xi^j_{(p)} \underset{def}{=} g^{jk}(x)p_k(x) \tag{3.1b}$$

The matrix $\left[g^{jk}(x)\right]$ is the inverse of the metric tensor:

$g^{ij}(x)g_{jk}(x) = \delta^i_k$

According to 3.1b, the right hand side of 3.1a is possible to be written in the form of an inner product:

$$\omega_p\left(\xi(x)\right) = \xi_{(p)}(x) \cdot \xi(x) = \left(\boldsymbol{x}_j \cdot \xi(x)\right)\xi^j_{(p)}(x) \tag{3.2}$$

Consequently, the value of any 1-form at any vector $\xi(x) \in T_x\boldsymbol{R}_0^3$ is calculated by the inner product of $\xi(x)$ with the vector field $\xi_{(p)}(x)$ specified by the particular 1-form.

We can easily verify that the set of the 1-forms defined on certain $T_x\boldsymbol{R}_0^3$ has the structure of a vector space (the addition-operation is the usual addition of two real functions and scalar multiplication, the multiplication of a real number with a real function); this space is called "**the dual space**" of $T_x\boldsymbol{R}_0^3$ [(7)].
Given that any 1-form is completely determined by a certain vector field, we infer that the 1-forms of $\boldsymbol{R}_0^3$ are dual to the vector fields of the same space.

Some special 1-forms

a) The 1-forms dx_j are defined by the relationships:

$$dx_j\left(\xi(x)\right) \underset{def}{=} \boldsymbol{x}_j \cdot \xi(x) = \boldsymbol{x}_j \cdot \boldsymbol{x}_k\, \xi^k(x) = g_{jk}\, \xi^k(x) \underset{def}{=} \xi_j(x) \tag{3.3}$$

Every 1-form is possible to be written as a linear combination of the forms dx_j as follows:

From 3.2 and 3.3 we result that:

$\omega_p\left(\xi(x)\right) = \xi^j_{(p)}(x) dx_j\left(\xi(x)\right)$

Hence:

$$\omega_p = \xi^j_{(p)}(x) dx_j \tag{3.4}$$

b) The 1-forms dx^j are defined according to the relation:

$$dx^j \underset{def}{=} g^{jk} dx_k \tag{3.5a}$$

$$dx^j\left(\xi(x)\right) = g^{jk} dx_k\left(\xi(x)\right) = g^{jk} g_{kn} \xi^n(x) = \xi^j(x) \tag{3.5b}$$

We result that the 1-forms dx^j return to the vector $\xi(x)$ its j-Cartesian coordinate.

Any 1-form can be expanded in a linear combination of the forms dx^j according to the expression:

$$\omega_p = g_{jk} \xi^j_{(p)}(x) dx^k \tag{3.6a}$$

From 3.6a we conclude that:

$$\omega_p\left(\xi(x)\right) = g_{jk} \xi^j_{(p)}(x) dx^k\left(\xi(x)\right) = g_{jk} \xi^j_{(p)}(x) \xi^k(x) \tag{3.6b}$$

Due to this property, the forms dx^j are called "basic forms" with respect to the Cartesian coordinates of the 3-dimensional Euclidean space.

c) **The differential** (or exterior derivative) of a real-valued differentiable function $f : \boldsymbol{R}^3_0 \to R$ is defined as the 1-form:

$$df\left(x\right) \underset{def}{=} \partial_j f\left(x\right) dx^j \tag{3.7a}$$

$\partial_j f\left(x\right) \underset{def}{=} \dfrac{\partial f\left(x\right)}{\partial x^j}$

The differential of a function *f* can have as domain any vector field defined on the Euclidean space. Let us assume the vector field defined by the tangent vectors of the curve:

$c : x_t = x(t)$

We express the tangent vectors of *c* in the form:

$\Delta x(t) = \dot{x}\left(t\right) \Delta t \in T_{x(t)} \boldsymbol{R}^3_0$, $\Delta t \in \boldsymbol{R}$ [1]

The previous relation defines the vector field $\Delta x(t)$ along the curve *c*: for each *t* the corresponding vector of the field is tangent to the curve *c* at its points *x*(*t*).

The action of the 1-form $df(x)\big|$ on the vectors $\Delta x = \Delta x(t)$ of the field returns the real values:

$$df(x)\big|_{\Delta x} = \partial_j f\left(x\right) dx^j(\Delta x) = \partial_j f\left(x\right) \Delta x^j \tag{3.7b}$$

The quantities $\Delta x^j = \dot{x}^j(t) \Delta t$ are the Cartesian coordinates of the vector field:

$\Delta x(t) = \boldsymbol{x}_j\, \Delta x^j(t) = \boldsymbol{x}_j\, \dot{x}^j(t) \Delta t$

[1] We often use the symbol $\Delta x = \dot{x}(t) \Delta t$ for a tangent vector. This symbolism is fruitful when we consider infinitesimal tangent vectors, i.e. when we let: $\Delta t \to 0$

Consider the 1-form: $\omega_p = p_j(x)dx^j$

If we can find a function $f(x)$ such that:

$\omega_p = df(x) = \partial_j f(x)dx^j$, $\omega_p(\Delta x) = df(x)\big|_{\Delta x} = \partial_j f(x)\Delta x^j$, $\Delta x \in T_x\boldsymbol{R}_0^3$

Then, we say that ω_p is an **exact 1-form**.

d) The **directional derivative** of the previous function f at x, along the vector $\Delta x \in T_x\boldsymbol{R}_0^3$ is defined by the relationship:

$$d_{\Delta x}f(x) = \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon\Delta x)-f(x)}{\varepsilon} \tag{3.8a}$$

An immediate consequence of this definition is the identity:

$$d_{\Delta x}f(x) = \partial_j f(x)\Delta x^j = df(x)\big|_{\Delta x} \tag{3.8b}$$

We imply that the directional derivative of $f(x)$ is identical with the differential of $f(x)$.

Remark: The definition 3.8a of the directional derivative is valid for the case of the 3-dimensional Euclidean space as long as both the points x of the space and the vectors Δx of the tangent spaces are determined by Cartesian triples. This is not generalized per se, in the case of more complicated structures, like the geometric surfaces that are to be studied in the forthcoming paragraphs, where the points and the tangent vectors are determined in completely different procedures.

Integration of a 1-form along a curve *c* of the Euclidean space

On a 3-dimensional Euclidean space, consider the curve $c: x_t = c(t)$, $t \in I \subset \boldsymbol{R}$ and the 1-form $\omega_p = p_j(x)dx^j$ expressed in Cartesian coordinates.

Let $\Delta x_t = \dot{c}(t)\Delta t \in T_{c(t)}\boldsymbol{R}_0^3$, $\Delta t \to 0$ be an infinitesimal tangent vector of c at its point $c(t)$.

The action of ω_p at the tangent vectors Δx_t of the curve c returns the infinitesimal real values:

$\omega_p(\Delta x_t) = p_j(c(t))\dot{c}^j(t)\Delta t$

We define the integral of ω_p along the curve c, among two points P_1, P_2 of the curve according to the relationship:

$$\int_{P_1}^{P_2}\omega_p = \int_{P_1}^{P_2}p_j(x)dx^j \underset{def}{=} \int_{t_1}^{t_2}p_j(c(t))dx^j\left(\dot{c}^j(t)\Delta t\right) = \int_{t_1}^{t_2}p_j(c(t))\dot{c}^j(t)dt \tag{3.9}$$

The points P_1, P_2 are lying on the curve c; they are determined by the triples $c(t_1)$, $c(t_2)$.

4. 2-forms

In this paragraph we are going to introduce the 2-forms of a 3-dimensional Euclidean space. The concept of the 2-forms is intimately related with the 1-forms. This relation will become clear by pondering on the result of the integration of a 1-form along the boundary of an elementary parallelogram of the Euclidean space. Eventually, we come to the idea of the wedge product of two 1-forms and then the definition of the 2-forms; our results will lead us to a formulation of the Stokes' theorem for the case of 1 and 2-forms determined on a 3-dimensional Euclidean space.

Consider the 1-form $\omega_p = p_j(x)dx^j$ and an elementary parallelogram, symbolized by $\Pi_x[\Delta_1 x, \Delta_2 x]$ of the 3-dimensional Euclidean space. In Cartesian coordinates, the vertex x of the parallelogram is determined by the triple $x=(x_1,x_2,x_3)$; its sides are determined by the

vectors $\Delta_1 x,\ \Delta_2 x \in T_x \boldsymbol{R}_0^3$ (figure 4.1). Our aim is to integrate ω_p along the boundary $\partial\pi_x[\Delta_1 x, \Delta_2 x]$ of the elementary parallelogram and evaluate the result when both sides tend to zero: $\Delta_1 x,\ \Delta_2 x \to 0$

The result of the integration is achieved by applying 3.9 and the mean value theorem [4], [6].

Remark: In the Euclidean or pseudo-Euclidean spaces the result of the **parallel displacement** of a vector $\Delta_1 x \in T_x \boldsymbol{R}_0^3$ from the point *x* to another *y*, is by definition a vector $\Delta_1 y \in T_y \boldsymbol{R}_0^3$ having identical **Cartesian** coordinates with the initial vector:

$\Delta_1 y^j = \Delta_1 x^j$

This condition is **not** general: we shall see that the parallel displacement is determined by a more complicated procedure in geometric surfaces or non-Euclidean spaces.

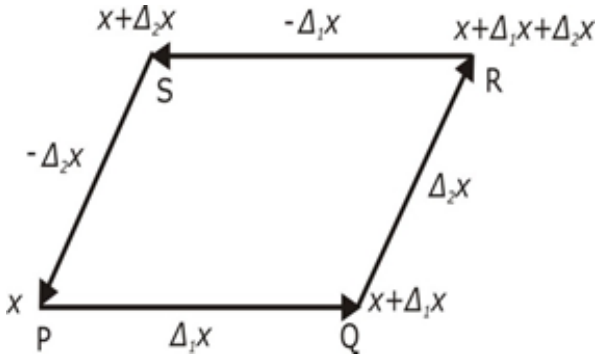

Figure 4.1: An elementary parallelogram of the Euclidean space. When both sides $\Delta_1 x,\ \Delta_2 x$ tend to zero, the parallelogram is shrinking to the point *P* determined by the triple $x=(x_1, x_2, x_3)$.

Let us now proceed to the integration of the 1-form ω_p along the boundary $\partial\pi_x[\Delta_1 x, \Delta_2 x]$ of the infinitesimal parallelogram:

$$\oint_{\partial\pi_x[\Delta_1 x, \Delta_2 x]} \omega_p = \oint_{\partial\pi_x[\Delta_1 x, \Delta_2 x]} p_j(x)\,dx^j$$

$$\oint_{\partial\pi_x[\Delta_1 x, \Delta_2 x]} \omega_p = \int_x^{x+\Delta_1 x} p_j(\bar{x})d\bar{x}^j + \int_{x+\Delta_1 x}^{x+\Delta_1 x+\Delta_2 x} p_j(\bar{x})d\bar{x}^j - \int_x^{x+\Delta_2 x} p_j(\bar{x})d\bar{x}^j - \int_{x+\Delta_2 x}^{x+\Delta_1 x+\Delta_2 x} p_j(\bar{x})d\bar{x}^j \approx$$

$$\approx p_j(x)\Delta_1 x^j + p_j(x+\Delta_1 x)\Delta_2 x^j - p_j(x)\Delta_2 x^j - p_j(x+\Delta_2 x)\Delta_1 x^j \approx$$

$$\approx \partial_k p_j(x)\Delta_1 x^k \Delta_2 x^j - \partial_k p_j(x)\Delta_2 x^k \Delta_1 x^j$$

By taking into account that $\Delta_1 x, \Delta_2 x \to 0$ we have applied the mean value theorem; we have expanded the functions $p_j(x+\Delta_1 x),\ p_j(x+\Delta_2 x)$ in Taylor series and have kept terms up to the second order with respect to $\Delta_1 x^j,\ \Delta_2 x^k$ [4], [5], [6].

The final result is expressed by the relationship:

$$\lim_{\Delta_1 x, \Delta_2 x \to 0} \oint_{\partial\pi_x[\Delta_1 x, \Delta_2 x]} \omega_p = \lim_{\Delta_1 x, \Delta_2 x \to 0} \oint_{\partial\pi_x[\Delta_1 x, \Delta_2 x]} p_j(x)dx^j = \partial_k p_j(x)\left(\Delta_1 x^k \Delta_2 x^j - \Delta_1 x^j \Delta_2 x^k\right) \quad (4.1)$$

We define the **wedge product** $dx^j \wedge dx^k$ of the basic 1-forms dx^j, dx^k (see paragraph 3) to be the bilinear antisymmetric form:

$$dx^j \wedge dx^k\left(\Delta_1 x, \Delta_2 x\right) \underset{def}{=} \Delta_1 x^j \Delta_2 x^k - \Delta_1 x^k \Delta_2 x^j \tag{4.2a}$$

$$dx^j \wedge dx^k\left(\Delta_1 x, \Delta_2 x\right) = \varepsilon^{jk}_{j_1 k_1} \Delta_1 x^{j_1} \Delta_2 x^{k_1} \tag{4.2b}$$

The antisymmetric symbol $\varepsilon^{jk}_{j_1 k_1}$ is defined as follows [5]:

a) The values of *j, k* are in the set {1,2,3}. If *j*=*k*, then: $\varepsilon^{jk}_{j_1 k_1} = 0$

b) The symbols j_1, k_1 take values in the set {*j, k*}. For *j*≠*k*, he value of $\varepsilon^{jk}_{j_1 k_1}$ equals to the parity of the permutation: $\begin{pmatrix} j, & k \\ j_1, & k_1 \end{pmatrix}$

We define a **2-form** σ as any linear combination of the wedge products of the basic 1-forms dx^i, *i*=1,2,3:

$$\sigma \underset{def}{=} h_{jk}(x) dx^j \wedge dx^k \tag{4.3}$$

The symbols $h_{jk}(x)$ stand for real functions defined on the Euclidean space.

The exterior derivative of a 1-form

The exterior derivative of a 1-form $\omega_p = p_j(x) dx^j$ is defined to be the 2-form:

$$d\omega_p \underset{def}{=} dp_j(x) \wedge dx^j = \partial_k p_j(x) dx^k \wedge dx^j \tag{4.4}$$

From 4.1-5 and for $\Delta_1 x, \Delta_2 x \to 0$ we imply that:

$$d\omega_p\left(\Delta_1 x, \Delta_2 x\right) = \oint_{\partial \pi_x[\Delta_1 x, \Delta_2 x]} \omega_p \tag{4.5}$$

Let us try to apply relation 4.5 to calculate the integral of the 1-form ω_p along a plane, closed curve *C*, which is the boundary of a compact and simply connected set [6] R_C:

$C \underset{def}{=} \partial R_C$

The set R_C is possible to be approximated by the union of a collection of infinitesimal parallelograms $\pi_x[\Delta_1 x, \Delta_2 x]$, $x \in R_C$ (figure 4.2). The integration of the 1-form ω_p is accomplished as follows:

$$\oint_{C=\partial R_C} \omega_p = \lim_{\pi[\Delta_1 x, \Delta_2 x] \to 0} \left\{ \sum_{\pi_x[\Delta_1 x, \Delta_2 x]} \left(\int_{\pi[\Delta_1 x, \Delta_2 x]} \omega_p \right) \right\} =$$

$$= \lim_{\pi[\Delta_1 x, \Delta_2 x] \to 0} \left\{ \sum_{\pi_x[\Delta_1 x, \Delta_2 x]} d\omega_p\left(\Delta_1 x, \Delta_2 x\right) \right\} \underset{def}{=} \int_{R_C} d\omega_p$$

$$\oint_{C=\partial R_C} \omega_p = \int_{R_C} d\omega_p \tag{4.6}$$

Relation 4.6 is a special case of the famous Stokes' theorem.

Proposition 4.1

The integral of an exact 1-form ω_p along any closed curve *c* equals to zero. Inversely: if the integral of a 1-form ω_p along any closed curve *c* equals to zero, then ω_p is an exact form.

Steps to the proof

If ω_p is exact, then there is a function *f*(*x*) such that: $\omega_p = df\left(x\right) = \partial_j f\left(x\right) dx^j$

We are going to verify that the exterior derivative of $\omega_p = df\left(x\right)$ is identically zero.

It holds:

$\partial_k \partial_j f(x) = \partial_j \partial_k f(x)$

$dx^k \wedge dx^j = -dx^j \wedge dx^k$

Hence:

$\partial_k \partial_j f(x) dx^k \wedge dx^j = -\partial_k \partial_j f(x) dx^j \wedge dx^k = -\partial_j \partial_k f(x) dx^k \wedge dx^j$

From which, we imply that:

$\partial_k \partial_j f(x) dx^k \wedge dx^j = 0$

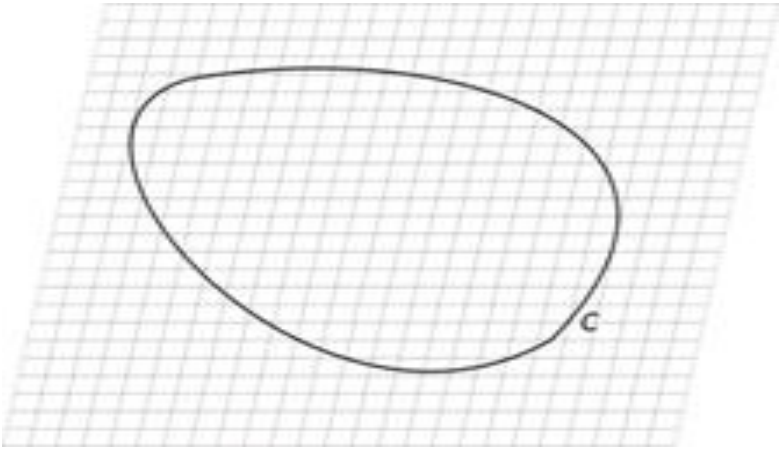

Figure 4.2: the simply connected, compact set R_C lying on a plane is approximated by the union of a collection of infinitesimal parallelograms.

The exterior derivative of $\omega_p = df(x)$ is:

$$d\omega_p = ddf(x) = d\left(\partial_j f(x) dx^j\right) = \partial_k \partial_j f(x) dx^k \wedge dx^j = 0 \qquad (4.7)$$

The proof of the direct proposition follows by applying the Stokes theorem (4.6), under the condition 4.7.

The truth of the inverse proposition is obtained by following the steps:

a) Given that the integral of a 1-form ω_p along any closed curve equals to zero, we infer that the integral $\int_a^x \omega_p \Big|_c$ does not depend on the curve joining the points *a* and *x*:

$\int_a^x \omega_p \Big|_c = \int_a^x \omega_p \Big|_{c'}$

b) Because of (a), the function $f(x) = \int_a^x \omega_p \Big|_c$ is well-defined.

c) By setting:

$x = c(t)$, $\Delta x = \dot{c}(t)\Delta t$, $\Delta t \to 0$

$c(t + \Delta t) \approx c(t) + \dot{c}(t)\Delta t$

We apply 3.9 and with the help of the mean value theorem, we show that:

$$d_{\Delta x} f(x) = \int_{c(t)=x}^{c(t+\Delta t)=x+\Delta x} \omega_p = \int_t^{t+\Delta t} p_j\left(c(\tau)\right)\dot{c}^j(\tau)d\tau \approx p_j\left(c(t)\right)\dot{c}^j(t)\Delta t = p_j(x)\Delta x^j = \omega_p(\Delta x)$$

We conclude that ω_p is an exact form.

5. **Coordinate-transformations**

In Cartesian coordinates, any point P of an abstract Euclidean space is determined by a unique triple (x^1,x^2,x^3) of real numbers. If we change the coordinate system to the polar or spherical one, the point P does not change but the values of its coordinates do. The same is true for any vector belonging to any tangent space of the Euclidean space. We say that the points of the space and the vectors of its tangent spaces are invariant under a coordinate-transformation; but their coordinates are not: their values depend on the choice of the coordinate system. In general, there are quantities defined on a geometric entity (like a Euclidean or non-Euclidean space or a surface), whose explicit form or their values are left invariant under any coordinate transformation. These properties that are independent of the used coordinate system reflect the intrinsic structure of the studied geometric entity. Our goal is to find out these invariant properties. Some of them are: the elementary length of a curve, the area and the volume forms, the Gaussian curvature.

First of all we define the group of the diffeomorphic transformations in a Euclidean space. Then we study the variations induced by any coordinate-transformation to the coordinates of the tangent vectors, the basis-vectors, the 1 and 2-forms. We continue by trying to find out some first invariant quantities and forms.

Let P be a point of the 3-dimensional Euclidean space and $x=\left(x^1,x^2,x^3\right)=\boldsymbol{x}_j x^j$ the representation of P in Cartesian coordinates (paragraph 1). Consider a coordinate-transformation $x^j=x^j\left(\bar{x}\right)$ where the triple $\bar{x}=\left(\bar{x}^1,\bar{x}^2,\bar{x}^3\right)$ represents the same point P in the new coordinate system. We assume that the transformation $x^j=x^j\left(\bar{x}\right)$ has the following properties:

a) It is one-to-one and onto
b) It is differentiable at least up to the second order, with respect to every argument $\bar{x}^j$, $j=1,2,3$
c) The inverse transformation $\bar{x}^j=\bar{x}^j\left(x\right)$ is differentiable at least up to the second order too

We call these transformations "**diffeomorphisms**". The set $\mathrm{Diff}\left(\boldsymbol{R}_0^3\right)$ of the diffeomorphisms of $\boldsymbol{R}_0^3$ equipped with the operation of the composition of transformations, forms a group. From now on, when we use the term "coordinate-transformation", we mean that this transformation is a diffeomorphism.

Assume that the points $P\in\boldsymbol{R}_0^3$ are determined by the Cartesian triples $x=(x^1,x^2,x^3)$ and let $x^j=x^j\left(\bar{x}\right)$ be any coordinate-transformation. We are going to examine the implications of this transformation to the tangent spaces $T_P\boldsymbol{R}_0^3$ of the Euclidean space.

We know that the elements of $T_P\boldsymbol{R}_0^3$ are the tangent vectors at P of every curve passing from P; let us consider a specific tangent vector: $\xi\in T_P\boldsymbol{R}_0^3$

How do the coordinates of ξ change, under the coordinate-transformation $x^j=x^j\left(\bar{x}\right)$ of the underlying space?

Given that $\xi\in T_P\boldsymbol{R}_0^3$ there is a curve c of $\boldsymbol{R}_0^3$ passing by the point P such that ξ is a tangent vector of c:

$c: x_t^{\,j}=x^j\left(t\right),\ x\left(0\right)=x \rightleftarrows P$

The Cartesian coordinates of ξ are given by the relationships:

$\xi^j = \dot{x}^j(0),\ j = 1,2,3$

The analytic expression of the same curve *c* in the new coordinate system $\bar{x}$ is determined by the set of the functions:

$\bar{x}_t{}^j = \bar{x}^j\left(x(t)\right),\ j = 1,2,3$

Hence, the coordinates of the vector ξ in the new coordinate system are calculated by the equations:

$$\bar{\xi}^j = \dot{\bar{x}}^j\left(x(t)\right)\Big|_{t=0} = \left.\frac{\partial \bar{x}^j}{\partial x^k}\right|_x \dot{x}^k(0) = \left.\frac{\partial \bar{x}^j}{\partial x^k}\right|_x \xi^k = \partial_k \bar{x}^j\, \xi^k$$

Or, in matrix form:

$$\begin{bmatrix}\bar{\xi}^1\\ \bar{\xi}^2\\ \bar{\xi}^3\end{bmatrix} = \begin{pmatrix}\partial_1\bar{x}^1 & \partial_2\bar{x}^1 & \partial_3\bar{x}^1\\ \partial_1\bar{x}^2 & \partial_2\bar{x}^2 & \partial_3\bar{x}^2\\ \partial_1\bar{x}^3 & \partial_2\bar{x}^3 & \partial_3\bar{x}^3\end{pmatrix}\begin{bmatrix}\xi^1\\ \xi^2\\ \xi^3\end{bmatrix}$$

We define the Jacobian matrices of the transformations $\bar{x}^j = \bar{x}^j\left(x\right),\ x^j = x^j\left(\bar{x}\right)$ by the relations:

$$R^j_k(x) \underset{def}{=} \left.\frac{\partial \bar{x}^j}{\partial x^k}\right|_x = \partial_k \bar{x}^j\Big|_x$$

$$\bar{R}^j_k(\bar{x}) \underset{def}{=} \left.\frac{\partial x^j}{\partial \bar{x}^k}\right|_x = \bar{\partial}_k x^j\Big|_x$$

$$\left[R^j_k\right] \underset{def}{=} \begin{bmatrix}R^1_1 & R^1_2 & R^1_3\\ R^2_1 & R^2_2 & R^2_3\\ R^3_1 & R^3_2 & R^3_3\end{bmatrix}$$

$$\left[\bar{R}^j_k\right] = \left[R^j_k\right]^{-1} = \frac{1}{\det\left[R^j_k\right]}\begin{bmatrix}D(R^1_1) & -D(R^2_1) & D(R^3_1)\\ -D(R^1_2) & D(R^2_2) & -D(R^3_2)\\ D(R^1_3) & -D(R^2_3) & D(R^3_3)\end{bmatrix}$$

$$D(R^2_1) \underset{def}{=} \det\begin{bmatrix}R^1_2 & R^1_3\\ R^3_2 & R^3_3\end{bmatrix},\ldots$$

It holds:

$$R^k_n \bar{R}^n_j = \frac{\partial \bar{x}^k}{\partial x^n}\frac{\partial x^n}{\partial \bar{x}^j} = \frac{\partial \bar{x}^k}{\partial \bar{x}^j} = \delta^k_j$$

Hence, under the coordinate-transformation $x^j = x^j\left(\bar{x}\right)$ of the underlying space $\boldsymbol{R}^3_0$ the coordinates of any vector $\xi \in T_p\boldsymbol{R}^3_0$ change according to the linear transformations:

$$\xi^k = \bar{R}^k_j\, \bar{\xi}^j \tag{5.1a}$$

$$\bar{\xi}^k = R^k_j\, \xi^j \tag{5.1b}$$

The coordinates of the vector ξ have been changed, but the vector ξ has not. On the other hand, the linearity of the equations 5.1a and b implies that the basis of every tangent space, in the new coordinate system has also been transformed.

How do the basis-elements of the tangent spaces transform, under the previous coordinate-transformation?

Let $\bar{\boldsymbol{x}}_j$ *j*=1,2,3 be the basis-elements of $T_p\boldsymbol{R}^3_0$ in the $\bar{x}$-coordinate system.

Each vector $\xi \in T_P\boldsymbol{R}_0^3$ is invariant under any change of the coordinates we are using to represent it. Hence, we have:

$$\xi = \bar{\boldsymbol{x}}_j \bar{\xi}^j = \boldsymbol{x}_\kappa \xi^\kappa \tag{5.2a}$$

$$\bar{\boldsymbol{x}}_j R_k^j \xi^k = \boldsymbol{x}_\kappa \xi^k$$

$$\left(\bar{\boldsymbol{x}}_j R_k^j - \boldsymbol{x}_k\right)\xi^k = 0$$

$$\boldsymbol{x}_k = \bar{\boldsymbol{x}}_j R_k^j \,,\ \bar{\boldsymbol{x}}_j = \boldsymbol{x}_k \bar{R}_j^k \tag{5.2b}$$

Example 5A

The polar coordinate system in the Euclidean plane

In this example, we apply the concepts and the geometric relations we have dealt up to now, for the case of the Euclidean plane: a) define the transformation from Cartesian to polar coordinates and its inverse, b) derive the Jacobian of these transformations, c) derive the corresponding coordinate-transformation on the tangent spaces of the Euclidean plane, d) find the explicit form of the polar basis-elements on the tangent spaces, e) calculate the matrix-elements of the metric tensor in polar coordinates.

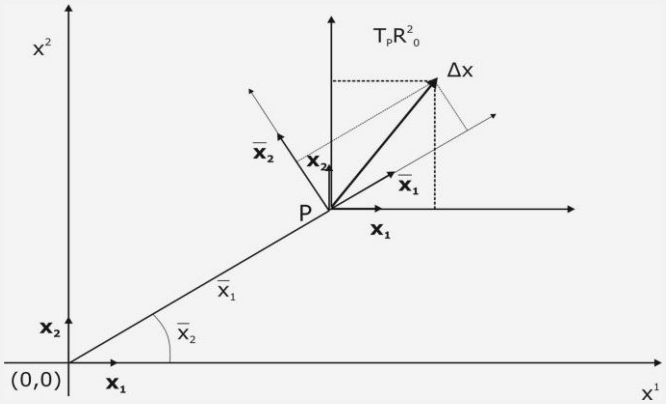

Figure 5.1: Polar coordinates in the Euclidean plane.

Let us consider the Euclidean plane $\boldsymbol{R}_0^2$ and a system of Cartesian coordinates x^j, j=1,2 in it. Any point $P \in \boldsymbol{R}_0^2$ is determined by its Cartesian coordinates x^1, x^2 (figure 5.1). We write:

$$P \rightleftarrows x = (x^1, x^2) = \boldsymbol{x}_1 x^1 + \boldsymbol{x}_2 x^2$$

$$\boldsymbol{x}_1 = (1,0),\ \boldsymbol{x}_2 = (0,1)$$

The polar coordinates ($\bar{x}^j$, $j = 1,2$) in $\boldsymbol{R}_0^2$ are determined by the transformation:

$$\begin{aligned} x^1 &= \bar{x}^1 \cos \bar{x}^2 \\ x^2 &= \bar{x}^1 \sin \bar{x}^2 \end{aligned} \tag{E5A.1a}$$

$\bar{x}^1 \in (0, +\infty)$, $\bar{x}^2 \in [0, 2\pi)$

The inverse transformation is defined everywhere in the plane, except the point (0,0); it is given by the equations:

$$\begin{aligned} \bar{x}^1 &= \left((x^1)^2 + (x^2)^2\right)^{1/2} \\ \bar{x}^2 &= \tan^{-1}\left(\frac{x^2}{x^1}\right) \end{aligned} \tag{E5A.1b}$$

Let $\Delta x \in T_P\boldsymbol{R}_0^2$ and $\Delta x^1, \Delta x^2$ its Cartesian coordinates:

$\Delta x = \boldsymbol{x}_1 \Delta x^1 + \boldsymbol{x}_2 \Delta x^2$

The polar coordinates of Δx are calculated by applying relations 5.1a:

$$\begin{aligned} \Delta x^1 &= \cos \bar{x}^2 \, \Delta \bar{x}^1 - \bar{x}^1 \sin \bar{x}^2 \, \Delta \bar{x}^2 \\ \Delta x^2 &= \sin \bar{x}^2 \, \Delta \bar{x}^1 + \bar{x}^1 \cos \bar{x}^2 \, \Delta \bar{x}^2 \end{aligned} \qquad \text{(E5A.2)}$$

By solving the system of equations E5A.2 for $\Delta \bar{x}^1, \Delta \bar{x}^2$ we find:

$$\begin{aligned} \Delta \bar{x}^1 &= \cos \bar{x}^2 \, \Delta x^1 + \sin \bar{x}^2 \, \Delta x^2 \\ \Delta \bar{x}^2 &= \frac{1}{\bar{x}^1}\left(-\sin \bar{x}^2 \, \Delta x^1 + \cos \bar{x}^2 \, \Delta x^2\right) \end{aligned} \qquad \text{(E5A.3)}$$

Notice that the explicit forms of the matrices $\left[\bar{R}_k^j\right], \left[R_k^j\right]$ in the case of our example are:

$$\left[\bar{R}_k^j\right] = \begin{pmatrix} \cos \bar{x}^2 & -\bar{x}^1 \sin \bar{x}^2 \\ \sin \bar{x}^2 & \bar{x}^1 \cos \bar{x}^2 \end{pmatrix}$$

$$\left[R_k^j\right] = \begin{pmatrix} \cos \bar{x}^2 & \sin \bar{x}^2 \\ -\frac{1}{\bar{x}^1} \sin \bar{x}^2 & \frac{1}{\bar{x}^1} \cos \bar{x}^2 \end{pmatrix}$$

It is well-known that the transformation $x^j = x^j\left(\bar{x}\right)$ is invertible in some neighborhood of any point $\bar{x}$ of its domain, if only the determinant of its Jacobian matrix $\left[\bar{R}_k^j\right]$ at $\bar{x}$ is non-zero [4], [6]. The determinant of $\left[\bar{R}_k^j\right]$ is:

$\det\left[\bar{R}_k^j\right] = \bar{x}^1$

We result that $x^j = x^j\left(\bar{x}\right)$ is invertible everywhere except the points (x^1,x^2) which satisfy the condition:

$\bar{x}^1 = \left((x^1)^2 + (x^2)^2\right)^{1/2} = 0$

We conclude that $x^j = x^j\left(\bar{x}\right)$ is invertible everywhere except the origin (0,0).

We are now ready to calculate the basis-elements $\bar{\boldsymbol{x}}_1$, $\bar{\boldsymbol{x}}_2$ of $T_P\boldsymbol{R}_0^2$ corresponding to the polar coordinate system:

Let $\Delta \bar{x}^1$, $\Delta \bar{x}^2$ the coordinates of the vector $\Delta x \in T_P\boldsymbol{R}_0^2$ with respect to the new basis.

We apply relations 5.2b and we find:

$\bar{\boldsymbol{x}}_1 = \boldsymbol{x}_1 \cos \bar{x}^2 + \boldsymbol{x}_2 \sin \bar{x}^2 = \left(\cos \bar{x}^2, \sin \bar{x}^2\right)$

$\bar{\boldsymbol{x}}_2 = -\boldsymbol{x}_1 \, \bar{x}^1 \sin \bar{x}^2 + \boldsymbol{x}_2 \, \bar{x}^1 \cos \bar{x}^2 = \bar{x}^1\left(-\sin \bar{x}^2, \cos \bar{x}^2\right)$

$\bar{\boldsymbol{x}}_1 = \frac{1}{\sqrt{(x^1)^2 + (x^2)^2}}\left(x^1, x^2\right)$

$\bar{\boldsymbol{x}}_2 = \left(-x^2, x^1\right)$

Verify that the vector $\Delta x \in T_P\boldsymbol{R}_0^3$ in polar coordinates is written:

$$\Delta x = \bar{\boldsymbol{x}}_1 \Delta \bar{x}^1 + \bar{\boldsymbol{x}}_2 \Delta \bar{x}^2 \qquad \text{(E5A.4)}$$

How shall we calculate the norm $|\Delta x|$ of Δx in polar coordinates?

From E5A.4 and the properties of the inner product, we have:

$|\Delta x|^2 = \langle \Delta x, \Delta x \rangle = \langle \bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_1 \rangle (\Delta \bar{x}^1)^2 + \langle \bar{\boldsymbol{x}}_2, \bar{\boldsymbol{x}}_2 \rangle (\Delta \bar{x}^2)^2 + 2 \langle \bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2 \rangle \Delta \bar{x}^1 \Delta \bar{x}^2$

We need the analytic expression of the metric tensor in polar coordinates; from the previous equations relating the polar with the "natural" (Cartesian) basis, we find:

$$\bar{g}_{11} = \langle \bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_1 \rangle = 1,\ \bar{g}_{12} = \bar{g}_{21} = \langle \bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2 \rangle = \langle \bar{\boldsymbol{x}}_2, \bar{\boldsymbol{x}}_1 \rangle = 0,\ \bar{g}_{22} = \langle \bar{\boldsymbol{x}}_2, \bar{\boldsymbol{x}}_2 \rangle = (\bar{x}^1)^2 \qquad \text{(E5A.5)}$$

Hence:

$$|\Delta x| = \sqrt{(\Delta \bar{x}^1)^2 + (\bar{x}^1)^2 (\Delta \bar{x}^2)^2}$$

Remarks:

a) The basis-elements $\bar{\boldsymbol{x}}_1$, $\bar{\boldsymbol{x}}_2$ change when we move from any point *P* to another point *Q* of the plane; i.e. when we move from the tangent space $T_P \boldsymbol{R}_0^3$ to another $T_Q \boldsymbol{R}_0^3$ of the Euclidean plane.

b) The basis $\{\bar{\boldsymbol{x}}_1, \bar{\boldsymbol{x}}_2\}$ is orthogonal but not unitary as long as the length of $\bar{\boldsymbol{x}}_2$ is not in general, equal to 1.

c) The analytic expression of the metric tensor has been changed in the new coordinates.

Invariant scalar fields defined on a 3-dimensional Euclidean space

We remind some points about the symbolism we have been using: *x* symbolizes a Cartesian triple determining the position of some point $P \in \boldsymbol{R}_0^3$ and $\Delta x = \dot{x}(0)\Delta t$ is the tangent vector of some curve $x_t = x(t)$ passing by *x*: $x(0) = x$

The same point *P* in another coordinate system is determined by another triple $\bar{x}$ which is related with *x* by a coordinate-transformation.

We are using the symbols Δx and $\Delta \bar{x}$ to symbolize the same tangent vector in the two coordinate systems, respectively. We have already emphasized that Δx and $\Delta \bar{x}$ are identical; they suggest that the same vector is expressed in two different basis elements and its coordinates are different in each system:

$$\Delta \bar{x} \equiv \Delta x = \bar{\boldsymbol{x}}_1 \Delta \bar{x}^1 + \bar{\boldsymbol{x}}_2 \Delta \bar{x}^2 = \boldsymbol{x}_1 \Delta x^1 + \boldsymbol{x}_2 \Delta x^2 \qquad (5.3)$$

Scalar fields on $\boldsymbol{R}_0^3$

Let us now consider a real function:

$F(x; \Delta x),\ \Delta x \in T_x \boldsymbol{R}_0^3$

We assume that $F(x; \Delta x)$ is defined on every tangent space $T_x \boldsymbol{R}_0^3$ of $\boldsymbol{R}_0^3$ but its form is possible to depend on the position *x* of the tangent space. We say that *F* defines a **scalar field** on the 3-dimensional Euclidean space.

For example, the cosine of the angle *u* of $\Delta \bar{x} \in T_{\bar{x}} \boldsymbol{R}_0^2$ relative to the basis vector $\bar{\boldsymbol{x}}_1$ in polar coordinates is calculated by the expression:

$$\cos u = \frac{\bar{\boldsymbol{x}}_1 \cdot \Delta \bar{x}}{|\bar{\boldsymbol{x}}_1||\Delta \bar{x}|} = \frac{\bar{\boldsymbol{x}}_1 \cdot (\bar{\boldsymbol{x}}_1 \Delta \bar{x}^1 + \bar{\boldsymbol{x}}_2 \Delta \bar{x}^2)}{\sqrt{\bar{g}_{11}} \sqrt{\bar{g}_{11} (\Delta \bar{x}^1)^2 + \bar{g}_{22} (\Delta \bar{x}^2)^2}}$$

According to relation E5A.5 of the Example 5A, we result that:

$$\cos u = \frac{\Delta \bar{x}^1}{\sqrt{(\Delta \bar{x}^1)^2 + (\bar{x}^1)^2 (\Delta \bar{x}^2)^2}}$$

We see that cos*u* is a function of $\Delta \bar{x} = \bar{\boldsymbol{x}}_1 \Delta \bar{x}^1 + \bar{\boldsymbol{x}}_2 \Delta \bar{x}^2$ but its value depends on $\bar{x}^1$ too; i.e. it depends on the position of the tangent space $T_{\bar{x}} \boldsymbol{R}_0^2$ in the Euclidean space.

We say that $F(x;\Delta x)$ is invariant under the coordinate-transformation $x^j = x^j(\bar{x})$ if only the following condition is satisfied:

$F(x;\Delta x) = F(\bar{x};\Delta\bar{x})$

For example, the function $F(\Delta x) = (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$ defined on the tangent spaces of $\boldsymbol{R}_0^3$ is invariant under the linear coordinate-transformation:

$x^1 = \bar{x}^1 \cos u - \bar{x}^2 \sin u$

$x^2 = \bar{x}^1 \sin u + \bar{x}^2 \cos u$

$x^3 = \bar{x}^3$

The coordinates of the tangent vectors are transformed according to the relations:

$\Delta x^1 = \Delta\bar{x}^1 \cos u - \Delta\bar{x}^2 \sin u$

$\Delta x^2 = \Delta\bar{x}^1 \sin u + \Delta\bar{x}^2 \cos u$

$\Delta x^3 = \Delta\bar{x}^3$

Hence:

$F(\Delta x) = (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 =$

$= \left(\Delta\bar{x}^1 \cos u - \Delta\bar{x}^2 \sin u\right)^2 + \left(\Delta\bar{x}^1 \sin u + \Delta\bar{x}^2 \cos u\right)^2 + \left(\Delta\bar{x}^3\right)^2 =$

$= (\Delta\bar{x}^1)^2 + (\Delta\bar{x}^2)^2 + (\Delta\bar{x}^3)^2 = F(\Delta\bar{x})$

Vectors

We have seen that any vector $\Delta x = \boldsymbol{x}_j \Delta x^j \in T_P\boldsymbol{R}_0^3$ is invariant under any coordinate-transformation of the underlying space. According to the transformation rules 5.1 and 5.2, although both the basis-elements $\boldsymbol{x}_j$ and the coordinates Δx^j change, the sum $\boldsymbol{x}_j \Delta x^j$ remains unaltered:

$\Delta x = \boldsymbol{x}_j \Delta x^j = \bar{\boldsymbol{x}}_k R_j^k \bar{R}_n^j \Delta\bar{x}^n = \bar{\boldsymbol{x}}_k \delta_n^k \Delta\bar{x}^n = \bar{\boldsymbol{x}}_k \Delta\bar{x}^k = \Delta\bar{x}$

The elementary length on a curve

The length Δs of Δx is also an invariant quantity; it has the same value regardless of the coordinate system in which the measurement is being accomplished:

$\Delta s = \sqrt{\Delta x \cdot \Delta x} = \sqrt{\boldsymbol{x}_j \cdot \boldsymbol{x}_k \Delta x^j \Delta x^k} =$

$= \sqrt{\bar{\boldsymbol{x}}_m \cdot \bar{\boldsymbol{x}}_n R_j^m R_k^n \bar{R}_p^j \bar{R}_q^k \Delta\bar{x}^p \Delta\bar{x}^q} =$

$= \sqrt{\bar{\boldsymbol{x}}_m \cdot \bar{\boldsymbol{x}}_n \Delta\bar{x}^m \Delta\bar{x}^n} = \sqrt{\Delta\bar{x} \cdot \Delta\bar{x}} = \Delta\bar{s}$

The inner product

The same is true for the inner product of any two vectors $\Delta_1 x, \Delta_2 x \in T_P\boldsymbol{R}_0^3$

We can easily verify that:

$\Delta_1 x \cdot \Delta_2 x = \Delta_1\bar{x} \cdot \Delta_2\bar{x}$

The invariance of the length and of the inner product under any coordinate-transformation results from the specific transformation rule of the metric tensor $[g_{jk}]$:

$$g_{jk} = \boldsymbol{x}_j \cdot \boldsymbol{x}_k = \bar{\boldsymbol{x}}_m \cdot \bar{\boldsymbol{x}}_n R_j^m R_k^n = \bar{g}_{mn} R_j^m R_k^n = R_j^m \bar{g}_{mn} R_k^n \qquad (5.4a)$$

Or, in matrix-form:

$$\left[g_{jk}\right] = \left[R_m^j\right]^T \left[\bar{g}_{mn}\right] \left[R_k^n\right] \qquad (5.4b)$$

Example 5B

The metric tensor in polar coordinates

In the present example, we derive the analytic expression of the metric tensor on the tangent spaces of the Euclidean plane, in polar coordinates. Then we verify that the norm of any tangent vector is invariant under the transformation from the Cartesian to polar coordinates.

In Example 5A, we found how the basis-elements $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and the coordinates Δx^1, Δx^2 of a vector $\Delta x = \boldsymbol{x}_j \Delta x^j \in T_P\boldsymbol{R}_0^2$ transform in polar coordinates; we also derived the analytic form of the Jacobian matrix of the corresponding coordinate-transformation.
In Cartesian coordinates the metric tensor $\left[g_{jk}\right]$ in every tangent space $T_P\boldsymbol{R}_0^2$ has the form:

$$\left[g_{jk}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{E5B.1}$$

By applying 5.4a and b we obtain the analytic expression of the metric tensor in polar coordinates:

$$\left[\bar{g}_{mn}\right] = \begin{pmatrix} 1 & 0 \\ 0 & (\bar{x}^1)^2 \end{pmatrix} \tag{E5B.2}$$

We confirm the invariance of the length of any vector $\Delta x = \boldsymbol{x}_j \Delta x^j = \bar{\boldsymbol{x}}_j \Delta \bar{x}^j \in T_P\boldsymbol{R}_0^2$ for this special case:

$$\left(\Delta s\right)^2 = g_{ij}\Delta x^i \Delta x^j = (\Delta x^1)^2 + (\Delta x^2)^2 =$$
$$= (\cos \bar{x}^2\, \Delta \bar{x}^1 - \bar{x}^1 \sin \bar{x}^2\, \Delta \bar{x}^2)^2 + (\sin \bar{x}^2\, \Delta \bar{x}^1 + \bar{x}^1 \cos \bar{x}^2\, \Delta \bar{x}^2)^2 =$$
$$= (\Delta \bar{x}^1)^2 + (x^1)^2 (\Delta \bar{x}^2)^2 = \bar{g}_{mn}\Delta \bar{x}^m \Delta \bar{x}^n = \left(\Delta \bar{s}\right)^2$$

With the help of the following proposition we are able to build a special type of scalar fields $F\left(x;\Delta x\right)$ which are invariant under any coordinate-transformation of the underlying Euclidean space.

Proposition 5.1
Consider the scalar field F_A defined by the analytic expression

$$F_A\left(x;\Delta x\right) = \Delta x \cdot \left(A_P(\Delta x)\right) \tag{5.5}$$

A_P is a linear map, with domain the tangent space $T_P\boldsymbol{R}_0^3$ and range into its-self. In the x-coordinate system the point P is determined by the triple x; in the coordinate system defined by the transformation $\bar{x}^j = \bar{x}^j\left(x\right)$ the same point P is determined by the triple $\bar{x}$ (see the previous section of the present paragraph).
The scalar field F_A defined by 5.5 is invariant under any coordinate-transformation $x^j = x^j\left(\bar{x}\right)$ of the underlying space; i.e.: $F_A\left(x;\Delta x\right) = F_A\left(\bar{x};\Delta \bar{x}\right)$

Steps to the proof
a) *How do the matrix of the linear map* $A_P : T_P\boldsymbol{R}_0^3 \to T_P\boldsymbol{R}_0^3$ *transform under the coordinate-transformation* $x^j = x^j\left(\bar{x}\right)$ *of the underlying Euclidean space?*
With respect to the x-coordinate system, the matrix $\left[a_j^k(x)\right]$ of A_P is defined by the action of A_P on the basis elements of the tangent space $T_P\boldsymbol{R}_0^3$

$$A_P(\boldsymbol{x}_j) = \boldsymbol{x}_k a_j^k(x) \tag{5.6a}$$

On the other hand the matrix $\left[a_{kj}(x)\right]$ is determined by the relation

$$a_{kj}(x) \underset{def}{=} \boldsymbol{x}_k \cdot A_P(\boldsymbol{x}_j) \tag{5.6b}$$

How are the matrices $\left[a_j^k\right], \left[a_{kj}\right]$ *related with each other?*

$$a_{kj} \underset{def}{=} \boldsymbol{x}_k \cdot A_P(\boldsymbol{x}_j) = \boldsymbol{x}_k \cdot \boldsymbol{x}_n a_j^n = g_{kn} a_j^n \tag{5.6c}$$

b) *How do the matrices* $\left[a_j^k\right], \left[a_{kj}\right]$ *transform under the coordinate-transformation* $x^j = x^j\left(\bar{x}\right)$ *of the underlying space?*

The coordinates of the vectors Δx, $\Delta x' = A_P(\Delta x)$ transform according to 5.1. Hence, by 5.6a we imply that:

$\Delta x'^k = a_j^k \Delta x^j$

$\bar{R}_j^k \Delta \bar{x}'^j = a_j^k \bar{R}_n^j \Delta \bar{x}^n$

$\Delta \bar{x}'^j = R_m^j a_n^m \bar{R}_k^n \Delta \bar{x}^k$

From the last equation we conclude that the matrix of the linear map A_P has been transformed according to the relationship:

$$\bar{a}_k^j(\bar{x}) = R_m^j \, a_n^m(x) \bar{R}_k^n \tag{5.7a}$$

The matrix $\left[\bar{a}_k^j(\bar{x})\right]$ is the matrix of A_P in the $\bar{x}$-coordinate system:

The matrix $\left[a_{kj}\right]$ has been also transformed. We symbolize $\left[\bar{a}_{kj}\right]$ the corresponding matrix in the $\bar{x}$-coordinate system. We calculate the matrix-elements of $\left[\bar{a}_{kj}\right]$ by using 5.3 and 5.6:

$$\bar{a}_{kj}(\bar{x}) = \bar{\boldsymbol{x}}_k \cdot A_P\left(\bar{\boldsymbol{x}}_j\right) = \boldsymbol{x}_n \cdot A_P\left(\boldsymbol{x}_m\right) \bar{R}_k^n \bar{R}_j^m = a_{nm}(x) \bar{R}_k^n \bar{R}_j^m \tag{5.7b}$$

c) In the x-coordinate system, the action of F_A at Δx returns:

$F_A(x;\Delta x) = \Delta x \cdot \left(A_P(\Delta x)\right) = a_{kj}(x) \Delta x^k \Delta x^j$

Under the coordinate-transformation $x^j = x^j\left(\bar{x}\right)$ the vectors Δx and $A_P(\Delta x)$ are invariant:

$\Delta x = \Delta \bar{x}$

$A_P\left(\Delta x\right) = A_P\left(\boldsymbol{x}_j\right) \Delta x^j = A_P\left(\bar{\boldsymbol{x}}_k\right) R_j^k \, \bar{R}_n^j \, \Delta \bar{x}^n = A_P\left(\bar{\boldsymbol{x}}_k\right) \Delta \bar{x}^k = A_P(\Delta \bar{x})$

Hence:

$F_A(x;\Delta x) = \Delta x \cdot \left(A_P(\Delta x)\right) = \Delta \bar{x} \cdot \left(A_P(\Delta \bar{x})\right) = F_A(\bar{x};\Delta \bar{x})$

We now derive the last relation by following a straightforward path:

$F_A(x;\Delta x) = \Delta x \cdot \left(A_P(\Delta x)\right) = a_{kj}(x) \Delta x^k \Delta x^j = \bar{a}_{mn}(\bar{x}) R_k^m R_j^n \bar{R}_p^k \bar{R}_q^j \Delta \bar{x}^p \Delta \bar{x}^q =$

$= \bar{a}_{mn}(\bar{x}) \Delta \bar{x}^m \Delta \bar{x}^n = F_A(\bar{x};\Delta \bar{x})$

We conclude that any scalar field F_A of the form 5.5 is invariant under any coordinate-transformation $x^j = x^j\left(\bar{x}\right)$ of the underlying space.

Scalar fields of the form: $F(x;\Delta x) = f_{ij}(x) \Delta x^i \Delta x^j$

A) Consider a scalar field defined by the expression:

$F(x;\Delta x) = f_{ij}(x) \Delta x^i \Delta x^j$

F is a scalar field defined on the tangent spaces $T_P \boldsymbol{R}_0^3$ of a 3-dimensional Euclidean space.

In the x-coordinate system the point P, is determined by the triple $x = \left(x^1, x^2, x^3\right)$

By applying a coordinate-transformation $x^j = x^j(\bar{x})$ we define the $\bar{x}$-coordinate system and the same point *P* is determined by the triple: $\bar{x} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$

The value of *F* for some $\Delta x \in T_P\boldsymbol{R}_0^3$ is not dependent on the choice of the coordinate system.
In the $\bar{x}$-coordinate system let us symbolize *F* by the analytic expression:
$\bar{F}(\bar{x};\Delta\bar{x}) = \bar{f}_{ij}(\bar{x})\Delta\bar{x}^i\Delta\bar{x}^j$
It is always true that:
$F(x;\Delta x) = \bar{F}(\bar{x};\Delta\bar{x})$
$\bar{f}_{mn}(\bar{x})\Delta\bar{x}^m\Delta\bar{x}^n = f_{ij}(x)\Delta x^i\Delta x^j$
$\bar{f}_{mn}(\bar{x})\Delta\bar{x}^m\Delta\bar{x}^n = f_{ij}(x)\bar{R}^i_m\bar{R}^i_n\Delta\bar{x}^m\Delta\bar{x}^n$
Hence:

$$\bar{f}_{mn}(\bar{x}) = f_{ij}(x)\bar{R}^i_m\bar{R}^i_n \tag{5.8a}$$

We say that $F(x;\Delta x)$ is invariant under the coordinate-transformation $x^j = x^j(\bar{x})$ if only it is true that:

$$F(x;\Delta x) = F(\bar{x};\Delta\bar{x}) = \bar{F}(\bar{x};\Delta\bar{x}) \tag{5.8b}$$

From the definition 5.8b we imply that $F(x;\Delta x)$ is invariant under the coordinate-transformation $x^j = x^j(\bar{x})$ if only:

$$\bar{f}_{mn}(\bar{x}) = f_{mn}(\bar{x}) = f_{ij}(x)\bar{R}^i_m\bar{R}^i_n \tag{5.8c}$$

B) 5.8c is a sufficient and necessary condition for $F(x;\Delta x)$ to be an invariant function under the mentioned coordinate-transformation.
Assume that the quantities $f_{ij}(x)$ in the analytic expression of $F(x;\Delta x)$ are constants; i.e. they are independent of *x*: $f_{ij}(x) = f_{ij} = \text{constant}$
In that case the real function $F(\Delta x) = f_{ij}\Delta x^i\Delta x^j$ is invariant under a **subgroup** of the diffeomorphic group $\text{Diff}(\boldsymbol{R}_0^3)$ of coordinate-transformations. According to 5.8c, the Jacobian matrices of these transformations are to be determined by the conditions

$$f_{ij} = R^k_i f_{kl} R^l_j \tag{5.9a}$$

Or, in matrix-form:

$$\left[f_{ij}\right] = \left[R^i_k\right]^T\left[f_{kl}\right]\left[R^l_j\right] \tag{5.9b}$$

In Appendix 1 we show how to create a subgroup of coordinate-transformations, which leave invariant scalar fields of specific type, defined on 2-dimensional Euclidean or pseudo-Euclidean spaces.

Euclidean coordinate-transformations in a 3-dimensional Euclidean space

In Cartesian coordinates, the matrix-elements of the metric tensor in every tangent space of a 3-dimensional Euclidean space are given by the relationships:
$g_{ij} = \boldsymbol{x}_i \cdot \boldsymbol{x}_j = \delta_{ij}$
The symbols $\boldsymbol{x}_i$, $j = 1,2,3$ stand, as usually, for the elements of the "natural" basis defined in paragraph 1.
The norm Δs of any vector $\Delta x = \boldsymbol{x}_j\Delta x^j \in T_P\boldsymbol{R}_0^3$ is calculated by the equation:

$$\Delta s^2 = g_{jk}\Delta x^j\Delta x^k = (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 \tag{5.10a}$$

Assume a coordinate-transformation $x^j = x^j(\bar{x})$ of the Euclidean space $\boldsymbol{R}_0^3$ which leaves **the analytic form** of 5.10a invariant. If such a coordinate-transformation exists, we should have:

$$\Delta s^2 = \bar{g}_{mn}\Delta\bar{x}^m\Delta\bar{x}^n = (\Delta\bar{x}^1)^2 + (\Delta\bar{x}^2)^2 + (\Delta\bar{x}^3)^2 \qquad (5.10b)$$

The coordinates of any tangent vector expressed in the two systems are related by the equations:

$\Delta\bar{x}^m = R_j^m \Delta x^j$

The matrix $\left[R_j^m\right]$ is the Jacobian of the transformation: $\bar{x}^j = \bar{x}^j(x)$

It is expressed by the relationship:

$\left[R_j^m\right] = \left[\partial_j \bar{x}^m\right]$

According to 5.10a and b, the matrices of the metric tensor in the two coordinate systems are identical to each-other:

$\left[\bar{g}_{ij}\right] = \left[g_{ij}\right]$

For any case of Euclidean or pseudo-Euclidean space, the metric tensor when expressed in Cartesian coordinates is independent of the position of the corresponding tangent space. Hence, to find the group of coordinate-transformations that leave the matrix of the metric tensor invariant, we have to apply equation 5.9a:

$g_{ij} = R_i^k g_{kl} R_j^l$, $g_{ij} = R_i^k R_{kj}$

$g_{ji}\bar{R}_m^i = \bar{R}_m^i R_i^k R_{kj}$, $\bar{R}_{jm} = \delta_m^k R_{kj}$

$$\bar{R}_{jm} = R_{mj} = R^T{}_{jm} \qquad (5.11a)$$

Or, in matrix form:

$$\left[R_{jk}\right]^{-1} = \left[R_{jk}\right]^T \qquad (5.11b)$$

$R_{ij} \underset{def}{=} g_{im} R_j^m$

From 5.11a, we obtain:

$g^{nj}\bar{R}_{jm} = g^{nj} R^T{}_{jm}$

$$\bar{R}_m^n = R^{Tn}{}_m = R_n^m \qquad (5.11c)$$

Or, in matrix form:

$$\left[R_k^i\right]^{-1} = \left[R_k^i\right]^T \qquad (5.11d)$$

Equations 5.11a, b, c and d hold for any case of constant metric tensor. Hence they are true for a Euclidean space too; i.e. for the metric tensor:

$\left[\bar{g}_{ij}\right] = \left[g_{ij}\right] = \left[\delta_{ij}\right]$

Or for a Minkowski space:

$$\left[\bar{g}_{jk}\right] = \left[g_{jk}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equations 5.11a and b say that the inverse of the Jacobian matrix $\left[R_{jk}\right]$ of the coordinate transformation that leave the metric tensor invariant, equals the transpose of the same matrix. The same is true for the matrix $\left[R_k^i\right]$ according to 5.11c and d.

From 5.11b we imply that:

$\left[R^T{}_{ij}\right]\left[R_{ij}\right] = I$

$\left|\det\left[R_{ij}\right]\right|^2 = 1$

The matrices $[R_{ij}]$ with determinant +1 are called **orthogonal** and the corresponding coordinate-transformations are called **isometries**. For the case of the Euclidean space $\boldsymbol{R}_0^3$ the isometries are the **Euclidean transformations** or **rigid motions**. For the case of the Minkowski space $\boldsymbol{R}_1^3$ the isometries are the **Lorentz transformations** (Appendix 1).

Example 5C

Linear maps on the tangent spaces of the Euclidean plane (I)

In the examples 5C and 5D we verify the results of the proposition 5.1 for the case of a specific linear map A_P defined on the tangent spaces of the Euclidean plane. We calculate the matrices $\left[a_j^k\right]$ and $\left[a_{kj}\right]$ of A_P in Cartesian and polar coordinates. We confirm that the scalar field $F_A(x;\Delta x) = \Delta x \cdot \left(A_P(\Delta x)\right)$ has the same value in the two coordinate systems.

Consider the linear map A_P defined on the tangent spaces $T_P\boldsymbol{R}_0^2$ of the Euclidean plane. Assume that in Cartesian coordinates, in every $T_P\boldsymbol{R}_0^2$ the matrix $\left[a_j^k\right]$ of A_P is determined by the equations:

$A_P(\boldsymbol{x}_1) = \boldsymbol{x}_2$, $A_P(\boldsymbol{x}_2) = -\boldsymbol{x}_1 + 2\boldsymbol{x}_2$

According to 5.6a, we imply that the matrix $\left[a_j^k\right]$ of A_P is given by the expression:

$$\left[a_j^k\right] = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$$

What about the matrix $\left[a_{kj}\right]$ *of* A_P?

The coordinate system we are working is Cartesian, so we have that:

$\boldsymbol{x}_j \cdot \boldsymbol{x}_k = \delta_{jk}$

Hence, by 5.6b we infer that the matrix $\left[a_{kj}\right]$ is given by the same form:

$$\left[a_{kj}\right] = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$$

We shall see in the following example that the two matrices are getting different forms when we are working in a polar coordinate system.

The linear map A_P transforms any vector $\Delta x \in T_P\boldsymbol{R}_0^2$ with coordinates Δx^1, Δx^2 to another vector $\Delta x'$ with coordinates $\Delta x'^1$, $\Delta x'^2$ with respect to the same basis.

The coordinates of the two vectors are related by the equations:

$$\Delta x' = A_P(\Delta x) = A_P(\boldsymbol{x}_j \Delta x^j) = A_P(\boldsymbol{x}_j)\Delta x^j = \boldsymbol{x}_k a_j^k \Delta x^j = \boldsymbol{x}_k \Delta x'^k$$

$$\Delta x'^k = a_j^k \Delta x^j \qquad \text{(E5C.1)}$$

Or, in matrix form:

$$\begin{pmatrix} \Delta x'^1 \\ \Delta x'^2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \Delta x^1 \\ \Delta x^2 \end{pmatrix}$$

Calculation of the scalar field F_A in Cartesian coordinates:

$F_A(x;\Delta x) = \Delta x \cdot \left(A_P(\Delta x)\right)$

$$F_A(x;\Delta x) = \Delta x \cdot \left(A_P(\Delta x)\right) = \boldsymbol{x}_j \cdot \boldsymbol{x}_k \, \Delta x^j a_n^k \Delta x^n = g_{jk} a_n^k \Delta x^j \Delta x^n =$$

$$= a_{jn} \Delta x^j \Delta x^n = -\Delta x^1 \Delta x^2 + \Delta x^2 \Delta x^1 + 2\Delta x^2 \Delta x^2 = 2\left(\Delta x^2\right)^2$$

$$F_A(x;\Delta x) = 2\left(\Delta x^2\right)^2 \qquad \text{(E5C.2)}$$

Example 5D

Linear maps on the tangent spaces of the Euclidean plane (II)

In this example we find the explicit form of the matrices $\left[a_j^k\right]$ and $\left[a_{kj}\right]$ corresponding to the linear map A_P of the Example 5C, in polar coordinates.

We have shown that in Cartesian coordinates, the matrices $\left[a_j^k\right]$ and $\left[a_{kj}\right]$ of A_P are identical:

$$\left[a_j^k\right] = \left[a_{kj}\right] = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$$

In a polar coordinate system $\left[a_j^k\right]$ and $\left[a_{kj}\right]$ transform respectively, to the matrices $\left[\bar{a}_k^j\right]$ and $\left[\bar{a}_{kj}\right]$ (relations 5.7a and b).

The Jacobian matrices $\left[R_k^i\right]$ and $\left[\bar{R}_k^i\right]$ of the transformations relating the Cartesian to the polar coordinates are to be found in Example 5A. After some calculations we find that:

$$\left[\bar{a}_j^k\right] = \begin{pmatrix} 2\left(\sin\bar{x}^2\right)^2 & \bar{x}^1\left(-1+\sin 2\bar{x}^2\right) \\ \dfrac{1}{\bar{x}^1}\left(1+\sin 2\bar{x}^2\right) & 2\left(\cos\bar{x}^2\right)^2 \end{pmatrix} \qquad \text{(E5D.1a)}$$

$$\left[\bar{a}_{kj}\right] = \begin{pmatrix} 2\left(\sin\bar{x}^2\right)^2 & \bar{x}^1\left(-1+\sin 2\bar{x}^2\right) \\ \bar{x}^1\left(1+\sin 2\bar{x}^2\right) & 2\left(\bar{x}^1\cos\bar{x}^2\right)^2 \end{pmatrix} \qquad \text{(E5D.1b)}$$

Given that the metric tensor of $T_P R_0^2$ in polar coordinates takes the form (see Example 5B, relation E5B.2):

$$\left[\bar{g}_{mn}\right] = \begin{pmatrix} 1 & 0 \\ 0 & (\bar{x}^1)^2 \end{pmatrix}$$

We are able to confirm the identity:

$\bar{a}_{kj} = \bar{g}_{kn}\bar{a}_j^n$

Remark: Notice that in polar coordinates:

a) The matrices $\left[\bar{a}_k^j\right]$ and $\left[\bar{a}_{kj}\right]$ are not equal

b) The matrices $\left[\bar{a}_k^j\right]$ and $\left[\bar{a}_{kj}\right]$ of the linear map $A \,/\, T_P\boldsymbol{R}_0^2$ are functions of $\bar{x}$

Calculation of the scalar field F_A in polar coordinates:

$$\bar{F}_A(\bar{x};\Delta\bar{x}) = \Delta\bar{x} \cdot \left(A_P(\Delta\bar{x})\right) = \bar{g}_{jk}\bar{a}_n^k \Delta\bar{x}^j \Delta\bar{x}^n = \bar{a}_{jn}\Delta\bar{x}^j\Delta\bar{x}^n$$

We use E5D.1b and E5A.3 (see Example 5A); after some tedious calculations we find that:

$$\bar{F}_A(\bar{x};\Delta\bar{x}) = 2\left(\sin\bar{x}^2\right)^2\left(\Delta\bar{x}^1\right)^2 + 2\bar{x}^1\sin 2\bar{x}^2\,\Delta\bar{x}^1\Delta\bar{x}^2 + 2\left(\bar{x}^1\cos\bar{x}^2\right)^2\left(\Delta\bar{x}^2\right)^2 =$$

$$= 2\left(\sin\bar{x}^2\right)^2\left(\cos\bar{x}^2\,\Delta x^1 + \sin\bar{x}^2\,\Delta x^2\right)^2 +$$

$$+2\bar{x}^1\sin 2\bar{x}^2\left(\cos\bar{x}^2\,\Delta x^1 + \sin\bar{x}^2\,\Delta x^2\right)\frac{1}{\bar{x}^1}\left(-\sin\bar{x}^2\,\Delta x^1 + \cos\bar{x}^2\,\Delta x^2\right) +$$

$$+2\left(\bar{x}^1\cos\bar{x}^2\right)^2\left(\frac{1}{\bar{x}^1}\right)^2\left(-\sin\bar{x}^2\,\Delta x^1+\cos\bar{x}^2\,\Delta x^2\right)^2=$$

$$=2\left(\Delta x^2\right)^2$$

The comparison of this result, with equation E5C.2 leads us to the expected conclusion:

$F_A(x;\Delta x)=\bar{F}_A(\bar{x};\Delta\bar{x})$

6. **Invariant forms under the group of the isometric transformations**

In the present paragraph we examine the changes in the analytic expression of the forms implied by a coordinate-transformation. The concepts "area" and "volume" in the 3-dimensional Euclidean or pseudo-Euclidean spaces are defined as the invariant 2-form and 3-form, respectively, under the group of the isometric coordinate-transformations.

Transformations of the 1-forms under a general diffeomorphic coordinate-transformation

Consider the diffeomorphic coordinate transformation $\bar{x}^j=\bar{x}^j(x)$ in a 3-dimensional Euclidean space.

As usually, we symbolize by x the triples that determine the points of the space in a Cartesian coordinate system; the matrix-elements of the Jacobian of the transformation are determined by the equations (paragraph 5):

$R^j_n=\partial_n\bar{x}^j$

We recall the properties of the 1-forms defined in paragraph 3 and verify that:

$dx^k(\Delta x)=\Delta x^k=\bar{R}^k_j\Delta\bar{x}^j=\bar{R}^k_j\,d\bar{x}^j(\Delta x)$

$\Delta x=\boldsymbol{x}_j\Delta x^j=\bar{\boldsymbol{x}}_j\Delta\bar{x}^j=\Delta\bar{x}\in T_P\boldsymbol{R}^3_0$

By definition, the 1-forms $d\bar{x}^k$, $k=1,2,3$ satisfy the relationships:

$d\bar{x}^k\left(\bar{\boldsymbol{x}}_j\Delta\bar{x}^j\right)=\Delta\bar{x}^k$

Hence, we have:

$$dx^k=\bar{R}^k_j d\bar{x}^j \tag{6.1a}$$

In the $\bar{x}$-coordinates the 1-forms $d\bar{x}_k$ stand for the transformed dx_k 1-forms:

$d\bar{x}_k\left(\Delta\bar{x}\right)=\bar{\boldsymbol{x}}_k\cdot\Delta\bar{x}=\bar{g}_{kn}\Delta\bar{x}^n$

We have:

$dx_j(\Delta x)=\boldsymbol{x}_j\cdot\Delta x=R^k_j\,\bar{\boldsymbol{x}}_k\cdot\Delta\bar{x}=R^k_j\,d\bar{x}_k\left(\Delta\bar{x}\right)$

Hence:

$$dx_j=R^k_j d\bar{x}_k \tag{6.1b}$$

$$\begin{aligned} dx_j&=g_{jk}dx^k=g_{jk}\bar{R}^k_n d\bar{x}^n=\bar{R}_{jn}d\bar{x}^n\\ d\bar{x}_j&=\bar{g}_{jk}d\bar{x}^k=g_{jk}R^k_n dx^n=R_{jn}dx^n\end{aligned} \tag{6.1c}$$

We define as **vector 1-form** $\vec{\omega}$ on $\boldsymbol{R}^3_0$ every linear map with domain any tangent space $T_x\boldsymbol{R}^3_0$, $x\in\boldsymbol{R}^3_0$ and range in itself, with analytic expression:

$\vec{\omega}=\boldsymbol{x}_j f^j_k(x)dx^k$

The action of $\vec{\omega}$ at any $\Delta x\in T_x\boldsymbol{R}^3_0$ returns the vector:

$\vec{\omega}(\Delta x)\in T_x\boldsymbol{R}^3_0:\ \vec{\omega}\left(\Delta x\right)=\boldsymbol{x}_j f^j_k(x)\Delta x^k$

How does the analytic expression of $\vec{\omega}$ change under a coordinate-transformation of the Euclidean space?

Let us write $\vec{\omega}$ in the $\bar{x}$-coordinates, as follows:

$$\vec{\bar{\omega}} = \bar{\boldsymbol{x}}_j \bar{f}_k^j(\bar{x}) d\bar{x}^k$$

The transformed vector 1-form $\vec{\bar{\omega}}$ is determined by the condition (see paragraph 5):

$$\vec{\omega}(\Delta x) = \vec{\bar{\omega}}(\Delta \bar{x}) \tag{6.2}$$

Relation 6.2 implies that the quantities $f_k^j(x)$ transform according to the rule:

$$\bar{f}_m^n(\bar{x}) = R_j^n f_k^j(x) \bar{R}_m^k \tag{6.3}$$

We say that $\vec{\omega}$ is **invariant under the group of Euclidean transformations** if only $f_k^j(x)$ satisfy the condition (see paragraph 5):

$$\bar{f}_k^j(\bar{x}) = f_k^j(\bar{x}) \tag{6.4a}$$

$$f_m^n(\bar{x}) = \bar{R}_j^n f_k^j(x) R_m^k \tag{6.4b}$$

Relation 6.4b holds for every orthogonal matrix $[R_k^j]$.

We define the identity-vector 1-form:

$$dx : T_x\boldsymbol{R}_0^3 \to T_x\boldsymbol{R}_0^3,\ dx \underset{def}{=} \boldsymbol{x}_k dx^k$$

It easily verified that:

$$dx(\Delta x) = \Delta x$$

The vector form dx is the identity map of $T_x\boldsymbol{R}_0^3$ on itself. For any coordinate system we can show that

$$dx = \boldsymbol{x}_k dx^k = \bar{\boldsymbol{x}}_j d\bar{x}^k = \mathrm{Id}$$

Obviously, the identity vector 1-form is a solution of 6.4b; consequently it is an invariant vector 1-form on the Euclidean space.

Now, let us find a less trivial invariant vector 1-form, for the 2-dimensional Euclidean space. Conjecture that there exists a vector 1-form with $f_k^j(x) = \text{constant}$, $j,k = 1,2$ that is invariant under the isometric transformations of the 2-dimensional Euclidean space.

Then the quantities $f_k^j(x)$ should satisfy the equations (6.4b):

$$f_m^n = \bar{R}_j^n f_k^j R_m^k$$

In matrix-form:

$$\left[f_k^j\right] = \bar{R}\left[f_k^j\right]R$$

In the case that R is an infinitesimal isometric transformation, we have (Appendix 2):

$$R = I + \delta\varphi\Omega$$

$$\left[f_k^j\right] = \left(I - \delta\varphi\Omega\right)\left[f_k^j\right]\left(I + \delta\varphi\Omega\right)$$

We keep terms up to the firs order:

$$\left[f_k^j\right] = \left[f_k^j\right] + \delta\varphi\left[\left[f_k^j\right], \Omega\right]$$

$$\left[\left[f_k^j\right], \Omega\right] = 0 \tag{6.4c}$$

The commutator is defined by the relationship:

$$\left[\left[f_k^j\right], \Omega\right] \underset{def}{=} \left[f_k^j\right]\Omega - \Omega\left[f_k^j\right]$$

According to Appendix 2:

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The solutions of equation 6.4c satisfy the conditions:

$f_2^2 = f_1^1 = a$, $f_1^2 = -f_2^1 = b$

It is a matter of some tedious calculations to verify that the following equation is true:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

For any isometric transformation with Jacobian matrix:

$$R = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

We conclude that the vector 1-form with the following analytic expression is invariant under the isometric transformations of the 2-dimensional Euclidean space:

$\vec{\omega} = \Omega(\boldsymbol{x}_1 b + \boldsymbol{x}_2 a)dx^1 + (\boldsymbol{x}_1 b + \boldsymbol{x}_2 a)dx^2$

The linear transformation Ω satisfies the conditions:

$\Omega(\boldsymbol{x}_1) = -\boldsymbol{x}_2$, $\Omega(\boldsymbol{x}_2) = \boldsymbol{x}_1$

Transformation of the 2-forms under a diffeomorphic coordinate-transformation
Invariant 2-forms under the group of the Euclidean transformations
Definition of the area-form

In Cartesian coordinates, a vector 2-form defined on the 3-dimensional Euclidean space has the analytic expression:

$\vec{\omega}^{(2)} \underset{def}{=} \boldsymbol{x}_j f_{kn}^j(x) dx^k \wedge dx^n$

The basic 1-forms dx^j, j=1,2,3 are defined on the tangent spaces of the Euclidean space (see the previous sections of the present paragraph).

Under a coordinate-transformation $\bar{x}^j = \bar{x}^j(x)$ the mentioned 2-form transforms to the following:

$\vec{\bar{\omega}}^{(2)} = \bar{\boldsymbol{x}}_m \bar{f}_{pq}^m(\bar{x}) d\bar{x}^p \wedge d\bar{x}^q$

By using the transformation rules for the basis-elements and for the 1-forms dx^j we find that the quantities $\bar{f}_{pq}^m(\bar{x})$ are related to the $f_{kn}^j(x)$ according to the identities:

$$\bar{f}_{pq}^m(\bar{x}) = R_j^m f_{kn}^j(x) \bar{R}_p^k \bar{R}_q^n \tag{6.5b}$$

We say that the vector 2-form $\vec{\omega}^{(2)}$ is invariant under the coordinate-transformation $\bar{x}^j = \bar{x}^j(x)$ if only the following conditions are satisfied (see relations 6.4a and b):

$\bar{f}_{pq}^m(\bar{x}) = f_{pq}^m(\bar{x}) = R_j^m f_{kn}^j(x) \bar{R}_p^k \bar{R}_q^n$

$$f_{pq}^m(\bar{x}) R_n^q = R_j^m f_{kn}^j(x) \bar{R}_p^k \tag{6.6}$$

Proposition 6.1

Consider the vector 2-form:

$$dA \underset{def}{=} \boldsymbol{x}_1 \cdot dx^2 \wedge dx^3 + \boldsymbol{x}_2 \cdot dx^3 \wedge dx^1 + \boldsymbol{x}_3 \cdot dx^1 \wedge dx^2 \tag{6.7a}$$

The form dA is invariant under the group of the Euclidean coordinate-transformations of the 3-dimensional Euclidean space (see paragraph 5).

Steps to the proof

According to 6.1a, b, c and the transformation properties of the Jacobian matrix $\left[R_k^j\right]$ referred in paragraph 5, we can verify the following identities:

$$d\bar{x}^2 \wedge d\bar{x}^3 = R_j^2 R_k^3 dx^j \wedge dx^k = \left(R_1^2 R_2^3 - R_2^2 R_1^3\right) dx^1 \wedge dx^2 + \ldots =$$
$$= D\left(R_3^1\right) dx^1 \wedge dx^2 - D\left(R_2^1\right) dx^3 \wedge dx^1 + D\left(R_1^1\right) dx^2 \wedge dx^3$$
$$d\bar{x}^2 \wedge d\bar{x}^3 = \det\left[R_k^j\right]\left(\bar{R}_1^3 dx^1 \wedge dx^2 + \bar{R}_1^2 dx^3 \wedge dx^1 + \bar{R}_1^1 dx^2 \wedge dx^3\right)$$

We have seen that for the group of the Euclidean coordinate-transformations, the determinant of the Jacobian matrix $\left[R_k^j\right]$ is +1. The matrix $\left[R_k^j\right]$ and its inverse $\left[\bar{R}_p^k\right]$ satisfy the relation $\bar{R}_j^k = R_k^j$ (see paragraph 5). Hence we have:

$$d\bar{A} = \bar{\boldsymbol{x}}_1 d\bar{x}^2 \wedge d\bar{x}^3 + \bar{\boldsymbol{x}}_2 d\bar{x}^3 \wedge d\bar{x}^1 + \bar{\boldsymbol{x}}_3 d\bar{x}^1 \wedge d\bar{x}^2 =$$
$$= \boldsymbol{x}_k \bar{R}_1^k \left(\bar{R}_1^3 dx^1 \wedge dx^2 + \bar{R}_1^2 dx^3 \wedge dx^1 + \bar{R}_1^1 dx^2 \wedge dx^3\right) +$$
$$+ \boldsymbol{x}_k \bar{R}_2^k \left(\bar{R}_2^3 dx^1 \wedge dx^2 + \bar{R}_2^2 dx^3 \wedge dx^1 + \bar{R}_2^1 dx^2 \wedge dx^3\right) +$$
$$+ \boldsymbol{x}_k \bar{R}_3^k \left(\bar{R}_3^3 dx^1 \wedge dx^2 + \bar{R}_3^2 dx^3 \wedge dx^1 + \bar{R}_3^1 dx^2 \wedge dx^3\right) =$$
$$= \boldsymbol{x}_1 \cdot \left[dx^1 \wedge dx^2 \cdot \left(\bar{R}_j^1 R_3^j\right) + dx^2 \wedge dx^3 \cdot \left(\bar{R}_j^1 R_1^j\right) + dx^3 \wedge dx^1 \cdot \left(\bar{R}_j^1 R_2^j\right)\right] + \ldots =$$
$$= \boldsymbol{x}_1 \cdot dx^2 \wedge dx^3 + \boldsymbol{x}_2 \cdot dx^3 \wedge dx^1 + \boldsymbol{x}_3 \cdot dx^1 \wedge dx^2 = dA$$

We conclude that *dA* is invariant under the group of the Euclidean coordinate-transformations of the 3-dimensional Euclidean space.

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Remark: It is easily confirmed that the invariant 2-form *dA* defined by 6.7a can be written:

$$dA = \frac{1}{2} \varepsilon^j{}_{kl} \boldsymbol{x}_j \, dx^k \wedge dx^l \tag{6.7b}$$

The antisymmetric symbol $\varepsilon^j{}_{kl}$ is defined in the Appendix 2: it takes the values +1 if the permutation (j k l) has positive parity, -1 if it has negative parity and it is equal to zero for any other case [5], [6].

In Appendices 1 and 2 we have shown that an infinitesimal orthogonal matrix *R* is written:

$R = I + \varepsilon\Omega$, $\varepsilon \to 0$

The matrix Ω is antisymmetric. For coordinate-transformations corresponding to infinitesimal orthogonal matrices, the conditions 6.6 take the form:

$$f_{lq}^m \Omega_n^q + f_{qn}^m \Omega_l^q = f_{ln}^q \Omega_q^m \tag{6.8a}$$

It is a matter of simple calculations to verify that if we set $f^j_{kl} = \frac{1}{2}\varepsilon^j{}_{kl}$ the equations 6.8a are been satisfied: the quantities $\frac{1}{2}\varepsilon^j{}_{kl}$ are solutions of the equations 6.8a.

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The 2-form *dA* which is invariant under the group of the Euclidean coordinate-transformations is defined as the **area-form** in the 3-dimensional Euclidean space.

Assume two infinitesimal vectors:

$\Delta_1 x$, $\Delta_2 x \in T_x \boldsymbol{R}_0^3$

The action of the area-form *dA* on them returns the number:

$$dA\left(\Delta_1 x, \Delta_2 x\right) = \frac{1}{2} \varepsilon^j{}_{kl} \cdot \boldsymbol{x}_j \cdot \Delta_1 x^k \wedge \Delta_2 x^l \tag{6.8b}$$

The quantity $dA(\Delta_1 x, \Delta_2 x)$ is defined as the **area of the infinitesimal parallelogram** $\pi_x[\Delta_1 x, \Delta_1 x]$ (see paragraph 4). The area of the infinitesimal parallelogram determines the area-element on the 3-dimensional Euclidean space.

3-forms

We define the 3-forms $\omega_h^{(3)}$ on a 3-dimensional Euclidean space as the antisymmetric tri-linear maps determined by the relation:

$$\omega_h^{(3)} \underset{def}{=} h_{ijk}(x) dx^i \wedge dx^j \wedge dx^k \tag{6.9a}$$

The quantities $h_{ijk}(x)$ are real functions defined on the Euclidean space.

Given that the indices *i, j, k* take values 1, 2 or 3, relation 6.9a takes the form:

$$\omega_h^{(3)} = h(x) dx^1 \wedge dx^2 \wedge dx^3 \tag{6.9b}$$

$h(x) = \varepsilon^{ijl} h_{ijk}(x)$

The action of $\omega_h^{(3)}$ at the vectors $\Delta_1 x, \Delta_2 x, \Delta_3 x \in T_x \boldsymbol{R}_0^3$ returns the value:

$$\omega_h^{(3)}(\Delta_1 x, \Delta_2 x, \Delta_3 x) = h(x) \varepsilon_{ijk} \Delta_1 x^i \Delta_2 x^j \Delta_3 x^k \tag{6.9c}$$

The value of the quantities ε_{ijk} , ε^{ijk} equals to zero if any two of the values of *i, j, k* are equal; for $i \neq j \neq k$ the value of ε_{ijk} , ε^{ijk} equals to the parity of the permutation: $\begin{pmatrix} 1 2 3 \\ i j k \end{pmatrix}$

Steps to the proof

The following identity holds:

$$dx^1 \wedge dx^2 \wedge dx^3 (\Delta_1 x, \Delta_2 x, \Delta_3 x) = \Delta_1 x^i \Delta_2 x^j \Delta_3 x^k dx^1 \wedge dx^2 \wedge dx^3 (\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k) \tag{6.9d}$$

$$dx^1 \wedge dx^2 \wedge dx^3 (\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k) = \varepsilon^{123}_{i\,j\,k} \underset{def}{=} \varepsilon_{ijk} \tag{6.9e}$$

Then, by combining 6.9b, d and e, we obtain 6.9c.

A relation between 2 and 3-forms - The Stokes theorem

In the present part of paragraph 6 we define the concept of an infinitesimal parallelepiped in the 3-dimensional Euclidean space and we integrate a 2-form on its boundary. By evaluating the result of the integration, we come to the definition of the exterior derivative of a 2-form. A formulation of the Stokes theorem is also resulted, that relates the 2 and 3-forms in a 3-dimensional Euclidean space.

Let $\Delta_1 x, \Delta_2 x, \Delta_3 x \in T_x \boldsymbol{R}_0^3$ be three linearly independent infinitesimal vectors. The point *x* and the mentioned vectors determine an infinitesimal parallelepiped $\pi_x[\Delta_1 x, \Delta_2 x, \Delta_3 x]$ with vertex at *x*; when $\Delta_1 x, \Delta_2 x, \Delta_3 x \to 0$ the parallelepiped degenerates to the point *x*.

Consider the 2-form:

$\omega_p^{(2)} = p_{jk}(x) dx^j \wedge dx^k$

Let us integrate it on the boundary $\partial \pi_x[\Delta_1 x, \Delta_2 x, \Delta_3 x]$ of the determined infinitesimal parallelogram (figure 6.1); by applying the mean value theorem (see paragraph 4) and keeping terms up to the third order with respect to the infinitesimal quantities, we obtain:

$$\oint_{\partial \pi_x[\Delta_1 x, \Delta_2 x, \Delta_3 x]} \omega_p^{(2)} = \oint_{\partial \pi_x[\Delta_1 x, \Delta_2 x, \Delta_3 x]} p_{jk} dx^j \wedge dx^k =$$

$$= -p_{jk}(x)(dx^j \wedge dx^k)(\Delta_1 x, \Delta_2 x) - p_{jk}(x)(dx^j \wedge dx^k)(\Delta_2 x, \Delta_3 x) - p_{jk}(x)(dx^j \wedge dx^k)(\Delta_3 x, \Delta_1 x) +$$

$$+p_{jk}\left(x+\Delta_3 x\right)\left(dx^j\wedge dx^k\right)\left(\Delta_1 x,\Delta_2 x\right)+p_{jk}\left(x+\Delta_1 x\right)\left(dx^j\wedge dx^k\right)\left(\Delta_2 x,\Delta_3 x\right)+$$
$$+p_{jk}\left(x+\Delta_2 x\right)\left(dx^j\wedge dx^k\right)\left(\Delta_3 x,\Delta_1 x\right)=$$
$$=-p_{jk}\left(x\right)\varepsilon^{jk}_{j'k'}\Delta_1 x^{j'}\Delta_2 x^{k'}-p_{jk}\left(x\right)\varepsilon^{jk}_{j'k'}\Delta_2 x^{j'}\Delta_3 x^{k'}-p_{jk}\left(x\right)\varepsilon^{jk}_{j'k'}\Delta_3 x^{j'}\Delta_1 x^{k'}+$$
$$+p_{jk}\left(x+\Delta_3 x\right)\varepsilon^{jk}_{j'k'}\Delta_1 x^{j'}\Delta_2 x^{k'}+p_{jk}\left(x+\Delta_1 x\right)\varepsilon^{jk}_{j'k'}\Delta_2 x^{j'}\Delta_3 x^{k'}+p_{jk}\left(x+\Delta_2 x\right)\varepsilon^{jk}_{j'k'}\Delta_3 x^{j'}\Delta_1 x^{k'}=$$
$$=\partial_i p_{jk}\left(x\right)\left[\varepsilon^{jk}_{j'k'}\Delta_1 x^{j'}\Delta_2 x^{k'}\Delta_3 x^{i}+\varepsilon^{jk}_{j'k'}\Delta_1 x^{i}\Delta_2 x^{j'}\Delta_3 x^{k'}+\varepsilon^{jk}_{j'k'}\Delta_1 x^{j'}\Delta_2 x^{i}\Delta_3 x^{k'}\right]$$

We expand the summations in the brackets of the last expression and we end up with the result:

$$\oint_{\partial n_x[\Delta_1 x,\Delta_2 x,\Delta_3 x]}\omega_p^{(2)}=\partial_i p_{jk}\left(x\right)\varepsilon^{ijk}_{i'j'k'}\Delta_1 x^{i'}\Delta_2 x^{j'}\Delta_3 x^{k'}=\partial_i p_{jk}\left(x\right)dx^i\wedge dx^j\wedge dx^k\left(\Delta_1 x,\Delta_2 x,\Delta_3 x\right)$$

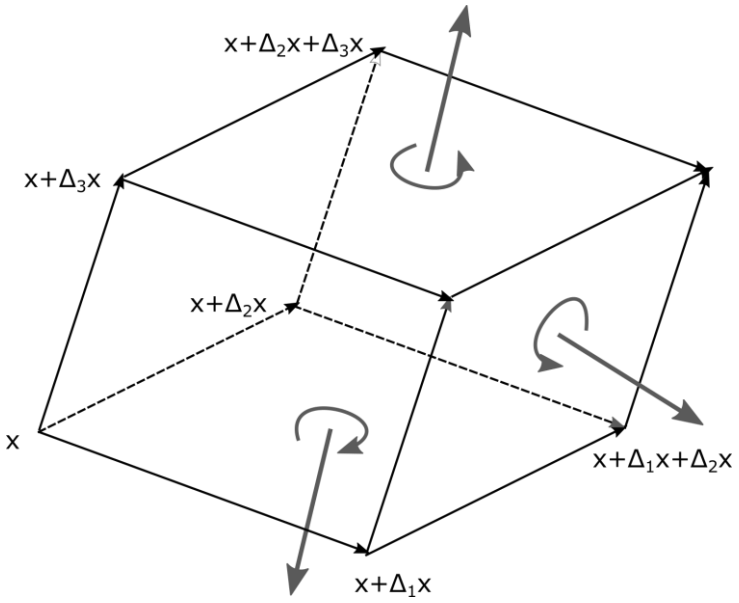

Figure 6.1: An infinitesimal parallelepiped in the three-dimensional Euclidean space. The parallelograms of its boundary are oriented to the exterior of the parallelepiped.

We define the **exterior derivative** of the 2-form $\omega_p^{(2)}=p_{jk}\left(x\right)dx^j\wedge dx^k$ to be the 3-form:

$$d\omega_p^{(2)}\underset{def}{=}d\left(p_{jk}dx^j\wedge dx^k\right)\underset{def}{=}\partial_i p_{jk}\left(x\right)dx^i\wedge dx^j\wedge dx^k$$

The results of the previous calculations are summarized to the derivation of the identity:

$$d\omega_p^{(2)}\left(\Delta_1 x,\Delta_2 x,\Delta_3 x\right)=\lim_{\Delta_1 x,\Delta_2 x,\Delta_3 x\to 0}\oint_{\partial n_x[\Delta_1 x,\Delta_2 x,\Delta_3 x]}\omega_p^{(2)}\qquad(6.10)$$

Assume a compact connected subset R_S of $\boldsymbol{R}_0^3$ with its boundary $S=\partial R_S$ being a closed surface in the underlying space. We are possible to approximate R_S by the union of a collection of infinitesimal parallelepipeds, in analogy to the procedure we followed in paragraph 4 (figure 4.2). Then, from 6.10 we obtain the following relation, which is another special case of the Stokes theorem:

$$\int_{R_S}d\omega_p^{(2)}=\oint_{\partial R_S}\omega_p^{(2)}\qquad(6.11)$$

Transformation of the 3-forms under a coordinate-transformation
Invariant 3-forms under the group of the Euclidean transformations

The volume element in the Euclidean spaces

We derive the transformation-rule of the 3-forms in a 3-dimensional Euclidean space, induced by any coordinate-transformation. Then we define the invariant 3-forms and we find out the subgroup of the coordinate-transformations that leave invariant the 3-forms; as a result, we are led to the definition of the volume-form and the elementary volume in a 3-dimensional Euclidean space.

Assume the 3-form $\omega_h^{(3)} = h(x)dx^1 \wedge dx^2 \wedge dx^3$ on $\boldsymbol{R}_0^3$ expressed in Cartesian coordinates.

Let $\bar{x}^j = \bar{x}^j(x)$ be any coordinate-transformation.

The 3-form $\omega_h^{(3)}$ transforms to the 3-form $\bar{\omega}_H^{(3)} = \bar{h}(\bar{x})d\bar{x}^1 \wedge d\bar{x}^2 \wedge d\bar{x}^3$ determined by the condition (see paragraph 5):

$$\omega_h^{(3)}(\Delta_1 x, \Delta_2 x, \Delta_3 x) = \bar{\omega}_{\bar{h}}^{(3)}(\Delta_1 \bar{x}, \Delta_2 \bar{x}, \Delta_3 \bar{x}) \tag{6.12}$$

The 1-forms dx^k transform according to the relations 6.1a; so we can write:

$\bar{\omega}_{\bar{h}}^{(3)} = \omega_h^{(3)} = h(x)dx^1 \wedge dx^2 \wedge dx^3 = h(x)\bar{R}_j^1 \bar{R}_k^2 \bar{R}_l^3 d\bar{x}^j \wedge d\bar{x}^k \wedge d\bar{x}^l =$

$= h(x)\bar{R}_j^1 \bar{R}_k^2 \bar{R}_l^3 \varepsilon_{123}^{jkl} d\bar{x}^1 \wedge d\bar{x}^2 \wedge d\bar{x}^3 = h(x)\det\bar{R}\, d\bar{x}^1 \wedge d\bar{x}^2 \wedge d\bar{x}^3 =$

$= \bar{h}(\bar{x})d\bar{x}^1 \wedge d\bar{x}^2 \wedge d\bar{x}^3$

The quantities $\bar{R}_k^j = \dfrac{\partial x^j}{\partial \bar{x}^k} \underset{def}{=} \bar{\partial}_k x^j$ are the matrix elements of the Jacobian matrix $\left[\bar{R}_j^k\right]$ of the transformation: $x^j = x^j(\bar{x})$

We conclude that under a general coordinate-transformation of $\boldsymbol{R}_0^3$ the 3-form $\omega_h^{(3)}$ transforms according to identity:

$\omega_h^{(3)} = h(x)dx^1 \wedge dx^2 \wedge dx^3 = h(x)\det\bar{R}\, d\bar{x}^1 \wedge d\bar{x}^2 \wedge d\bar{x}^3 = \bar{h}(\bar{x})d\bar{x}^1 \wedge d\bar{x}^2 \wedge d\bar{x}^3 = \bar{\omega}_{\bar{h}}^{(3)}$

$$\bar{h}(\bar{x}) = h(x)\det\bar{R} \tag{6.13}$$

When a 3-form of $\boldsymbol{R}_0^3$ is said to be invariant under a coordinate-transformation?

The 3-form $\omega_h^{(3)} = h(x)dx^1 \wedge dx^2 \wedge dx^3$ is invariant under a coordinate-transformation if only it is transformed to the form:

$\bar{\omega}_H^{(3)} = h(\bar{x})d\bar{x}^1 \wedge d\bar{x}^2 \wedge d\bar{x}^3$

I.e. $\omega_h^{(3)}$ is invariant under a coordinate-transformation if it is true that

$$h(\bar{x}) = h(x)\det\bar{R} \tag{6.14}$$

We have seen that the determinant of the Jacobian matrix $\bar{R} = \left[\bar{R}_j^k\right]$ corresponding to any Euclidean coordinate-transformation in the Euclidean space $\boldsymbol{R}_0^3$ equals to the unity:

$\det\bar{R} = 1$

Hence in the Euclidean space $\boldsymbol{R}_0^3$ the 3-forms which are invariant under the group of the **Euclidean coordinate-transformations** of $\boldsymbol{R}_0^3$ are the forms that satisfy the condition:

$h(\bar{x}) = h(x) = h = \text{constant}$

By choosing the appropriate system of units, we set h=1 and we define the **volume-form** in $\boldsymbol{R}_0^3$ to be the 3-form:

$$\Omega = dx^1 \wedge dx^2 \wedge dx^3 \tag{6.15a}$$

The action of the volume-form Ω at the infinitesimal vectors $\Delta_1 x, \Delta_2 x, \Delta_3 x \in T_x \boldsymbol{R}_0^3$ returns the **volume-element** of the infinitesimal parallelepiped $\pi_x[\Delta_1 x, \Delta_2 x, \Delta_3 x]$ (see the previous section of the present paragraph).
It holds that:

$$\Omega\left(\Delta_1 x, \Delta_2 x, \Delta_3 x\right) = \varepsilon_{ijk} \Delta_1 x^i \Delta_2 x^j \Delta_3 x^k \tag{6.15b}$$

Invariant 2 and 3-forms in the 3-dimensional pseudo-Euclidean space

In the preceding sections of paragraph 6, we defined the area-form and the volume-form of the 3-dimensional Euclidean space, as the forms which are invariant under the group of the Euclidean coordinate transformations. We found out that these forms are given respectively by the expressions (relations: 6.7b, 6.15a):

$$dA = \frac{1}{2} \varepsilon_{kl}^j \boldsymbol{x}_j dx^k \wedge dx^l$$

$$\Omega = dx^1 \wedge dx^2 \wedge dx^3$$

We are going to repeat the same task for the case of a 3-dimensional Minkowski space: we proceed to derive the 2- and 3-forms of the Minkowski space $\boldsymbol{R}_1^3$ which are invariant under the Lorentz transformations. We end up to the definition of the area-form and the volume-form in a 3-dimensional Minkowski space.

We know that the isometric coordinate-transformations of the Minkowski space $\boldsymbol{R}_1^3$ are the Lorentz transformations; *which 2-form on* $\boldsymbol{R}_1^3$ *is invariant under the Lorentz coordinate-transformations? How do we define the area-form* in a 3-dimensional Minkowski space*?*
We try to answer relying on the condition 6.6: the general sufficient and necessary condition that an invariant vector 2-form $\vec{\omega}^{(2)} \underset{def}{=} \boldsymbol{x}_j f_{kn}^j(x) dx^k \wedge dx^n$ should satisfy is expressed by the equation:

$$f_{pq}^m(\bar{x}) R_n^q = R_j^m f_{kn}^j(x) \bar{R}_p^k$$

We restrict our investigation to forms that: (a) are invariant under the group of the Lorentz transformations and (b) they have the same expression at any tangent space; i.e. in Cartesian coordinates, the quantities f_{kn}^j are independent of *x*:

$$\vec{\omega}^{(2)} = \boldsymbol{x}_j f_{kn}^j dx^k \wedge dx^n$$

Hence, in order that $\vec{\omega}^{(2)}$ be invariant under the group of the Lorentz transformations, the quantities f_{kn}^j should be solutions of the equations:

$$f_{pq}^m R_n^q = R_j^m f_{kn}^j \bar{R}_p^k \tag{6.16}$$

We are trying to solve the system of the equations 6.16; in order to simplify them we assume an infinitesimal Lorentz transformation. The corresponding Jacobian matrix $R = \left[R_k^j(x)\right]$ is infinitesimal (see Appendix 1) and takes the form:

$$R = I + \varepsilon\Omega,\ \varepsilon \to 0$$

Regardless of the form of the metric tensor $g = \left[g_{ij}(x)\right]$ the Jacobian matrix *R* of an isometric coordinate-transformation $x^j = x^j(\bar{x})$ satisfies the condition (see 5.9a and b)

$$g = R^T g R \tag{6.17a}$$

Hence, for the infinitesimal isometric coordinate-transformation, we obtain the following relationships:

$g = (I + \varepsilon\Omega)^T g (I + \varepsilon\Omega)$

$g\,\Omega + \Omega^T g = 0$

$$g_{ij}\Omega^j_k + \Omega^{Tn}_i g_{nk} = 0 \qquad (6.17b)$$

For the case of the Minkowski space $\boldsymbol{R}^3_1$ the metric tensor in Cartesian coordinates, is determined by the matrix:

$$[g] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We substitute in 6.17b and we solve the ensued equations for the unknown matrix elements of Ω-matrix; we result that:

$\Omega^1_1 = \Omega^2_2 = \Omega^3_3 = 0$

$\Omega^1_2 + \Omega^2_1 = 0$

$\Omega^1_3 - \Omega^3_1 = 0$

$\Omega^2_3 - \Omega^3_2 = 0$

$$\Omega = \begin{pmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & \omega^1 \\ \omega^2 & \omega^1 & 0 \end{pmatrix} \qquad (6.18)$$

Now, we apply 6.16 for the case of the infinitesimal transformations $R = I + \varepsilon\Omega$ and we are led to the equations:

$$f^m_{pq}\Omega^q_n + \Omega^k_p f^m_{kn} = \Omega^m_j f^j_{pn} \qquad (6.19)$$

We presume that the invariant vector 2-form *da* for the 3-dimensional Minkowski space is given by an expression similar to the area-form *dA* of the 3-dimensional Euclidean space; we write:

$da = f^1_{jk}\boldsymbol{x}_1 \cdot dx^j \wedge dx^k + f^2_{lm}\boldsymbol{x}_2 \cdot dx^l \wedge dx^m + f^3_{pq}\boldsymbol{x}_3 \cdot dx^p \wedge dx^q$

We then obtain the following solution of the equations 6.19:

$f^1_{11} = f^1_{12} = f^1_{21} = f^1_{13} = f^1_{31} = f^2_{12} = f^2_{21} = f^2_{32} = f^2_{23} = f^2_{22} = f^3_{13} = f^3_{31} = f^3_{32} = f^3_{23} = f^3_{33} = 0$

$f^1_{23} = f^2_{31} = -f^3_{12} = f^3_{21} = -f^1_{32} = -f^2_{13} = \lambda$

For a certain system of units we set $\lambda = 1$ and we come to the following assertion, which we are going to prove:

The subsequent 2-form is invariant under the Lorentz coordinate-transformations of a 3-dimensional Minkowski space:

$$da = \boldsymbol{x}_1 \cdot dx^2 \wedge dx^3 + \boldsymbol{x}_2 \cdot dx^3 \wedge dx^1 - \boldsymbol{x}_3 \cdot dx^1 \wedge dx^2 \qquad (6.20)$$

Steps to the proof

Assume a Lorentz coordinate-transformation $x^j = x^j(\bar{x})$ with Jacobian matrix: $R = \left[R^j_k(x)\right]$

We use the properties of the matrix $R = \left[R^j_k(x)\right]$ referred in paragraph 5 and we obtain the consequent identities (see: proof of the proposition 6.1):

$dx^2 \wedge dx^3 = \bar{R}^2_j \bar{R}^3_k d\bar{x}^j \wedge d\bar{x}^k = \left(\bar{R}^2_1\bar{R}^3_2 - \bar{R}^2_2\bar{R}^3_1\right) d\bar{x}^1 \wedge d\bar{x}^2 + \ldots =$

$= D\left(\bar{R}^1_3\right) d\bar{x}^1 \wedge d\bar{x}^2 - D\left(\bar{R}^1_2\right) d\bar{x}^3 \wedge d\bar{x}^1 + D\left(\bar{R}^1_1\right) d\bar{x}^2 \wedge d\bar{x}^3$

$dx^2 \wedge dx^3 = \det\bar{R}\left(R^3_1 d\bar{x}^1 \wedge d\bar{x}^2 + R^2_1 d\bar{x}^3 \wedge d\bar{x}^1 + R^1_1 d\bar{x}^2 \wedge d\bar{x}^3\right)$

Hence, the 2-form *da* is transformed as follows:

$$da = \boldsymbol{x}_1 \cdot dx^2 \wedge dx^3 + \boldsymbol{x}_2 \cdot dx^3 \wedge dx^1 - \boldsymbol{x}_3 \cdot dx^1 \wedge dx^2 =$$
$$= \bar{\boldsymbol{x}}_k R_1^k \left(R_1^3 d\bar{x}^1 \wedge d\bar{x}^2 + R_1^1 d\bar{x}^2 \wedge d\bar{x}^3 + R_1^2 d\bar{x}^3 \wedge d\bar{x}^1 \right) +$$
$$+ \bar{\boldsymbol{x}}_k R_2^k \left(R_2^3 d\bar{x}^1 \wedge d\bar{x}^2 + R_2^1 d\bar{x}^2 \wedge d\bar{x}^3 + R_2^2 d\bar{x}^3 \wedge d\bar{x}^1 \right) -$$
$$- \bar{\boldsymbol{x}}_k R_3^k \left(R_3^3 d\bar{x}^1 \wedge d\bar{x}^2 + R_3^1 d\bar{x}^2 \wedge d\bar{x}^3 + R_3^2 d\bar{x}^3 \wedge d\bar{x}^1 \right) =$$
$$= \bar{\boldsymbol{x}}_1 \cdot \left[\left(R_1^1 R_1^3 + R_2^1 R_2^3 - R_3^1 R_3^3 \right) d\bar{x}^1 \wedge d\bar{x}^2 + \left(R_1^2 R_1^3 + R_2^2 R_2^3 - R_3^2 R_3^3 \right) d\bar{x}^2 \wedge d\bar{x}^3 + \left(R_1^3 R_1^3 + R_2^3 R_2^3 - R_3^3 R_3^3 \right) d\bar{x}^3 \wedge d\bar{x}^1 \right] +$$
$$+ \bar{\boldsymbol{x}}_2 \cdot \left[\left(R_1^2 R_1^3 + R_2^2 R_2^3 - R_3^3 R_3^3 \right) d\bar{x}^1 \wedge d\bar{x}^2 + \left(R_1^2 R_1^1 + R_2^2 R_2^1 - R_3^2 R_3^1 \right) d\bar{x}^2 \wedge d\bar{x}^3 + \left(R_1^2 R_1^2 + R_2^2 R_2^3 - R_3^2 R_3^2 \right) d\bar{x}^3 \wedge d\bar{x}^1 \right] +$$
$$+ \bar{\boldsymbol{x}}_3 \cdot \left[\left(R_1^3 R_1^3 + R_2^3 R_2^3 - R_3^3 R_3^3 \right) d\bar{x}^1 \wedge d\bar{x}^2 + \left(R_1^3 R_1^1 + R_2^3 R_2^1 - R_3^3 R_3^1 \right) d\bar{x}^2 \wedge d\bar{x}^3 + \left(R_1^3 R_1^2 + R_2^3 R_2^3 - R_3^3 R_3^2 \right) d\bar{x}^3 \wedge d\bar{x}^1 \right]$$

Under the Lorentz coordinate-transformation $x^j = x^j(\bar{x})$ the metric tensor of each tangent space $T_x \boldsymbol{R}_1^3$ is invariant:

$$g_{ij} = g_{nm} R_i^n R_j^m$$
$$g_{ij} \bar{R}_m^j = g_{nm} R_i^n$$
$$g_{ij} \bar{R}_m^j = g_{mn} R_i^n$$

From which, we imply that:

$$R_1^1 = \bar{R}_1^1, R_2^1 = \bar{R}_1^2, R_3^1 = -\bar{R}_1^3, R_1^2 = \bar{R}_2^1, R_2^2 = \bar{R}_2^2, R_3^2 = -\bar{R}_2^3, R_1^3 = -\bar{R}_3^1, R_2^3 = -\bar{R}_3^2, R_3^3 = \bar{R}_3^3$$

We apply these properties of the matrix $\left[R_n^m \right]$ to the previous analytic expression of *da* and we find that:

$$da = \bar{\boldsymbol{x}}_1 \cdot \left[\bar{R}_1^j R_j^3 d\bar{x}^1 \wedge d\bar{x}^2 + \bar{R}_1^j R_j^1 d\bar{x}^2 \wedge d\bar{x}^3 + \bar{R}_1^j R_j^2 d\bar{x}^3 \wedge d\bar{x}^1 \right] +$$
$$+ \bar{\boldsymbol{x}}_2 \cdot \left[\bar{R}_2^j R_j^3 d\bar{x}^1 \wedge d\bar{x}^2 + \bar{R}_2^j R_j^1 d\bar{x}^2 \wedge d\bar{x}^3 + \bar{R}_2^j R_j^2 d\bar{x}^3 \wedge d\bar{x}^1 \right] +$$
$$+ \bar{\boldsymbol{x}}_3 \cdot \left[\bar{R}_3^j R_j^3 d\bar{x}^1 \wedge d\bar{x}^2 + \bar{R}_3^j R_j^1 d\bar{x}^2 \wedge d\bar{x}^3 + \bar{R}_3^j R_j^2 d\bar{x}^3 \wedge d\bar{x}^1 \right] =$$
$$= \bar{\boldsymbol{x}}_1 \cdot d\bar{x}^2 \wedge d\bar{x}^3 + \bar{\boldsymbol{x}}_2 \cdot d\bar{x}^3 \wedge d\bar{x}^1 + \bar{\boldsymbol{x}}_3 \cdot d\bar{x}^1 \wedge d\bar{x}^2 = d\bar{a}$$

•

The invariant vector 2-form *da* is called the **area-form** in the Minkowski space $\boldsymbol{R}_1^3$

To find the **volume-form** in $\boldsymbol{R}_1^3$ we follow a treatment analogous to that for the case of the 3-dimensional Euclidean space. We can easily verify that the 3-form of $\boldsymbol{R}_1^3$ which is invariant under the Lorentz coordinate-transformation is given by the expression:

$$\Omega = dx^1 \wedge dx^2 \wedge dx^3$$

The 3-form Ω is the volume-form in the 3-dimensional Minkowski space.

The action of the volume-form Ω at the infinitesimal vectors $\Delta_1 x, \Delta_2 x, \Delta_3 x \in T_x \boldsymbol{R}_1^3$ expresses the **volume-element** of an infinitesimal parallelepiped $\pi_x[\Delta_1 x, \Delta_2 x, \Delta_3 x]$ and is calculated by the relation:

$$\Omega(\Delta_1 x, \Delta_2 x, \Delta_3 x) = \varepsilon_{ijk} \Delta_1 x^i \Delta_2 x^j \Delta_3 x^k$$

Chapter 2

Key concepts

Simple surfaces - Curves on a surface - Tangent spaces of a surface S - Basis-vectors of the tangent spaces of S - The metric tensor on the tangent spaces of S - 1 and 2-forms defined on the tangent spaces of S - Integration of an 1-form on a curve of S - Exterior derivative of an 1-form - The Stokes' theorem - Vector Fields on a surface - Parameter transformations - Invariant 2-forms on S - The area-form on S - The geometric surface - Connections in the geometric surface - Infinitesimal displacement of a vector on a surface with respect to a connection - Covariant differentiation of a vector field - Connection compatible with the metric tensor - Parallel displacement of a vector - Curvature of a geometric surface - Geodesic curves on a geometric surface - Frame fields - Connection forms - Geodesic curvature of a curve on a geometric surface - The Gauss-Bonnet theorem

7. Geometric features of a surface

In this and the following paragraphs 8 and 9 we study the geometric features of a surface immersed in an underlying 3-dimensional Euclidean space. We borrow the main concepts from the discussed geometry of the Euclidean spaces and we develop the geometry of a surface by using the same tools and treading on a similar reasoning path. The goal is to become acquainted with the main geometric concepts we have introduced up to now by applying them to the study of a surface; our perspective is to get a deeper level on this subject which will be achieved by the paragraph 10 and then.

Simple surfaces in Euclidean or pseudo-Euclidean spaces

We define a simple surface [(1), (2), (3)] (we call it simply: "surface") S in a Euclidean or pseudo-Euclidean 3-dimensional space a set of points P_u which, in Cartesian coordinates, is determined by the vector-function:

$$S:\begin{cases} x^1 = x_S{}^1(u^1,u^2) \\ x^2 = x_S{}^2(u^1,u^2) \\ x^3 = x_S{}^3(u^1,u^2) \end{cases}$$

In brief, we symbolize:

$S: x = x_S(u)$

The ordered pairs $u=(u^1,u^2)$ run an open set B of the plane $\boldsymbol{R}^2$.

The map $x_S : B \to x_S(B) \subset \boldsymbol{R}_0^3$ has the following properties:

a) It is 1-1 and differentiable at least up to the second order.

b) The vector-functions $\partial_1 x_S \underset{def}{=} \frac{\partial x_S}{\partial u^1}$, $\partial_2 x_S \underset{def}{=} \frac{\partial x_S}{\partial u^2}$ are linearly independent throughout the domain B.

Remarks:

a) From now on, the Latin indices will run in the set {1,2,3} and the Greek indices in the set {1,2}.

b) In simple surfaces there is a 1-1 correspondence between the points P_u of the surface and the couples $u=(u^1,u^2)$ of the domain B. The geometric properties of a surface S arise from the geometric structure of the underlying Euclidean or pseudo-Euclidean space and the analytic expression of the surface.

c) In Cartesian coordinates, a surface of revolution [(1), (2)] S_f immersed in a Euclidean space, is given by the analytic expression:

$$S_f : \begin{cases} x^1 = f(u^2)\cos u^1 \\ x^2 = f(u^2)\sin u^1 \\ x^3 = u^2 \end{cases} \tag{7.1}$$

The domain B of the parameters $u=(u^1,u^2)$ is the set

$$u = \left(u^1, u^2\right) \in B \underset{def}{=} \bigcup_{n\in Z} \left[2n\pi, 2(n+1)\pi\right) \otimes \left(-b, b\right)$$

The real function $f(u^2)$ is differentiable at least up to the second order.

Curves on a surface

Any curve lying on the surface $S : x^j = x_S{}^j\left(u^1, u^2\right)$ is the image of some curve $a : u^\mu = a^\mu(t)$, $\mu = 1,2$ lying in the domain B of S (the parameter t runs an interval I of the real numbers $\boldsymbol{R}$).

We symbolize the image of a on S with the corresponding capital letter A. The analytic expression of the curve A is determined by the relation:

$$A = x_S \circ a : A(t) \underset{def}{=} \left(x_S \circ a\right)(t) = x_S\left(a(t)\right)$$

We frequently use the symbolism:

$$A(t) = \left(A^1(t), A^2(t), A^3(t)\right) = \left(x^1(t), x^2(t), x^3(t)\right)$$

$$A = x_S \circ a : x^j(t) = x_S{}^j\left(a^1(t), a^2(t)\right)$$

$$A = x_S \circ a : \begin{cases} x^1(t) = x_S{}^1\left(a^1(t), a^2(t)\right) \\ x^2(t) = x_S{}^2\left(a^1(t), a^2(t)\right) \\ x^3(t) = x_S{}^3\left(a^1(t), a^2(t)\right) \end{cases} \tag{7.2}$$

Remark: *Curves on a surface of revolution*

If our surface is a surface of revolution (S_f), the analytic expression of the curve A on S_f is:

$$A = x_{S_f} \circ a : \begin{cases} x^1(t) = f\left(a^2(t)\right)\cos\left(a^1(t)\right) \\ x^2(t) = f(a^2(t))\sin\left(a^1(t)\right) \\ x^3(t) = a^2(t) \end{cases} \tag{7.3}$$

Tangent spaces of a surface *S*

Consider a surface S immersed in the space the 3-dimensional Euclidean space determined in Cartesian coordinates, by the analytic expression:

$$S : x^j = x_S{}^j\left(u^1, u^2\right)$$

We define the tangent space of S at an arbitrary point P_u of S as the set of the vectors $\xi \in T_P\boldsymbol{R}_0^3$ which are tangent to some curve of S passing by P_u.

We symbolize the tangent space of S at any of its point P_u, with the alternative symbols:

$$T_{P_u}S,\ T_{x_S(u)}S,\ T_uS$$

For example, let $x_S \circ a : x^j(t) = x_S{}^j\left(a^1(t), a^2(t)\right)$ be a curve on S passing by P_u, such that:

$$a^1(0) = u^1,\ a^2(0) = u^2$$

Then:

a) The point P_u is determined by the triple:

$$x_S \circ a : x(0) = x_S\left(a^1(0), a^2(0)\right) = x_S\left(u^1, u^2\right)$$

b) The tangent vector at P_u is calculated by the equation:

$$\xi = \frac{d}{dt} x_S\left(a^1(t), a^2(t)\right)\Big|_{t=0}$$

c) It holds: $\xi \in T_{P_u}S \subset T_{P_u}\boldsymbol{R}_0^3$

The tangent space $T_{P_u}S$ contains all the tangent vectors of the curves of S passing by P_u, at their common point P_u.

By following "the steps to the proof" of the proposition 2.1 - paragraph 2, we can show that the tangent spaces of S are vector spaces. Each vector space $T_{P_u}S$ is a subspace of the corresponding tangent space $T_{P_u}\boldsymbol{R}_0^3$ of the underlying space at the same point.

Basis-elements of the tangent spaces of a surface *S*

Let $\xi \in T_uS$ be a tangent vector to the surface S. Then there is a curve c of the parameters' space such that its image-curve $C = x_S \circ c$ on S satisfies the following properties:

$$C = x_S \circ c : x(t) = x_S\left(c^1(t), c^2(t)\right),\ c(0) = u = \left(u^1, u^2\right)$$

$$\xi = \dot{x}(0) = \frac{\partial x_S}{\partial u^\mu}\Big|_u \dot{c}^\mu(0) \tag{7.4a}$$

From 7.4a we imply that any $\xi \in T_uS$ is written as a linear combination of the basis-elements:

$$e_\mu(u) = \frac{\partial x_S(u)}{\partial u^\mu},\ \mu = 1,2$$

$$\xi = e_\mu(u)\dot{c}^\mu(0) \tag{7.4b}$$

We conclude that:

a) The coordinates of any vector $\xi \in T_uS$ with respect to the basis-elements e_1(u), e_2(u) are the coordinates of the tangent vector of the corresponding curve in the parameters' space:

$$\xi = e_\mu(u)\xi^\mu = e_\mu(u)\dot{c}^\mu(0),\ \xi^\mu = \dot{c}^\mu(0) \tag{7.4c}$$

b) The tangent spaces T_uS, $u \in B \subseteq \boldsymbol{R}^2$ are two-dimensional vector spaces; we frequently call them "**tangent planes**" of S.

c) The basis-elements e_1(u), e_2(u) are tangents at t=0 of the surface-curves:

$$x_{(1)}(t) = x_S\left(u^1 + t, u^2\right)$$

$$x_{(2)}(t) = x_S\left(u^1, u^2 + t\right)$$

$$e_1(u) = \frac{d}{dt} x_S\left(u^1 + \mathrm{t}, u^2\right)\Big|_{t=0} = \frac{\partial x_S\left(u^1, u^2\right)}{\partial u^1} \tag{7.5a}$$

$$e_2(u) = \frac{d}{dt} x_S\left(u^1, u^2 + t\right)\Big|_{t=0} = \frac{\partial x_S\left(u^1, u^2\right)}{\partial u^2} \tag{7.5b}$$

d) The basis-elements e_1(u), e_2(u) are being expressed as linear combinations of the natural basis $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$ of the underlying Euclidean space:

$$e_1(u) = \boldsymbol{x}_j \frac{\partial x_S{}^j(u)}{\partial u^1} \tag{7.6a}$$

$$e_2(u) = \boldsymbol{x}_j \frac{\partial x_S{}^j(u)}{\partial u^2} \tag{7.6b}$$

Remark: For the case of the surface of revolution S_f (relation 7.1) we find that:

$$e_1(u) = -f(u^2)\sin u^1 \cdot \boldsymbol{x}_1 + f(u^2)\cos u^1 \cdot \boldsymbol{x}_2 \qquad (7.6c)$$

$$e_2(u) = f'(u^2)\cos u^1 \cdot \boldsymbol{x}_1 + f'(u^2)\sin u^1 \cdot \boldsymbol{x}_2 + \boldsymbol{x}_3 \qquad (7.6d)$$

We symbolize: $f'(u^2) \underset{def}{=} \dfrac{df(u^2)}{du^2}$

Vector fields on a surface *S*

We define as **vector field** $\xi(u)$ on a surface *S* any vector function determined on *S* which at every point P_u of *S* returns a vector $\xi(u) \in T_uS$ (see paragraph 2).

An example of a vector field is given by the expressions 7.6c and d: the basis-elements e_1(u), e_2(u) are defined for every point P_u of *S* and belong to the tangent space T_uS.

We assume that the vector fields we are going to deal with are differentiable functions of both the parameters u_1, u_2, at least up to the second order.

The metric tensor of the tangent planes of the surface *S*

The Euclidean metric of the underlying space induces a metric in the tangent planes of the surface *S*, as follows:

Consider the surface *S* determined in Cartesian coordinates by the functions:

$x^j = x_S{}^j(u^1, u^2),\ j = 1,2,3$

Let $\Delta x_{(q)}$, $q = 1,2,3\ldots$ symbolize tangent vectors of *S* belonging to the tangent plane T_uS.

For some $\Delta x_{(1)} \in T_uS$ there is a curve $u_{(1)}(t) = \left(u_{(1)}{}^1(t), u_{(1)}{}^2(t)\right)$ in the parameters' space *B* passing by *u*: $u_{(1)}(0) = u = (u^1, u^2)$ with tangent vector at *u* the vector:

$\Delta u_{(1)} = \left(\Delta_{(1)}u^1, \Delta_{(1)}u^2\right) = \left(\dot{u}_{(1)}{}^1(0), \dot{u}_{(2)}{}^2(0)\right)\Delta t,\ \Delta t \in \boldsymbol{R}$

The corresponding tangent vector of *S* at $x(u)=x_S(u)$ is determined by applying 7.4b:

$$\Delta_{(1)}x = e_\mu(u)\Delta_{(1)}u^\mu \qquad (7.7a)$$

$$\Delta_{(1)}x = e_\mu(u)\dot{u}_{(1)}{}^\mu(0)\Delta t \qquad (7.7b)$$

The vector $\Delta_{(1)}x$ is the tangent vector at $x_S(u)$ of the curve of *S* determined by the functions:

$x^j(t) = x_S{}^j\left(u_{(1)}{}^1(t), u_{(1)}{}^2(t)\right)$

The basis-elements of T_uS are expressed, according to 7.6a and b, as linear combinations of the natural basis of the tangent space $T_{x_S(u)}\boldsymbol{R}_0^3$ of the underline Euclidean space. Hence it is legitimate to calculate the Euclidean inner products of them according to the relationships:

$$e_\mu(u) \cdot e_\nu(u) = \boldsymbol{x}_j \cdot \boldsymbol{x}_k \frac{\partial x_S{}^j(u)}{\partial u^\mu}\frac{\partial x_S{}^k(u)}{\partial u^\nu}$$

Let us now consider any other vector: $\Delta_{(2)}x = e_\nu(u)\Delta_{(2)}u^\nu \in T_uS$

We symbolize the inner product of $\Delta_{(1)}x, \Delta_{(2)}x \in T_uS$ with the symbol $\langle \Delta_{(1)}x, \Delta_{(2)}x \rangle$ and we define it by the relation:

$\langle \Delta_{(1)}x, \Delta_{(2)}x \rangle = e_\mu(u) \cdot e_\nu(u)\Delta_{(1)}u^\mu\Delta_{(2)}u^\nu = g_{\mu\nu}(u)\Delta_{(1)}u^\mu\Delta_{(2)}u^\nu$

The matrix-elements of the metric tensor $g(u) = \left[g_{\mu\nu}(\mathrm{u})\right]$ defined on the tangent space T_uS, with respect to the basis-elements e_1(u), e_2(u) are determined by the equation:

$$g_{\mu\nu}(u) = e_\mu(u)\cdot e_\nu(u) = \boldsymbol{x}_j \cdot \boldsymbol{x}_k \frac{\partial x_S{}^j(u)}{\partial u^\mu}\frac{\partial x_S{}^k(u)}{\partial u^\nu} \tag{7.8a}$$

We have pointed out that the points of the underlying space $\boldsymbol{R}_0^3$ are determined in Cartesian coordinates; hence:

$\boldsymbol{x}_j \cdot \boldsymbol{x}_k = \delta_{jk}$

We conclude that the matrix-elements of the metric tensor $g(u)$ are calculated by the expression:

$$g_{\mu\nu}(u) = \sum_{j=1}^{3} \frac{\partial x_S{}^j(u)}{\partial u^\mu}\frac{\partial x_S{}^j(u)}{\partial u^\nu} \tag{7.8b}$$

Remarks:

A) We outline a proof that the metric tensor given by 7.8b is an **invertible** 2x2 matrix for any value of u.

This is due to the restriction (b) in the definition of a simple surface: the linear independence of the row-vectors:

$\partial_1 x_S = \left(\partial_1 x_S{}^1, \partial_1 x_S{}^2, \partial_1 x_S{}^3\right)$, $\partial_2 x_S = \left(\partial_2 x_S{}^1, \partial_2 x_S{}^2, \partial_2 x_S{}^3\right)$

So, if we assume that $g(u)$ is not invertible for some point u=(u^1, u^2) of the parameters' space, then the row vectors of $g(u)$ should be linearly dependent i.e. there is some real number $\lambda \neq 0$ such that:

$(g_{11}, g_{12}) = \lambda(g_{21}, g_{22})$

Then, according to 7.8b, we imply that (notice that the product of quantities with the same indices is to be interpreted as a summation with respect to the repeated index):

$\left(\partial_1 x_S{}^j \partial_1 x_S{}^j, \partial_1 x_S{}^k \partial_2 x_S{}^k\right) = \lambda\left(\partial_2 x_S{}^j \partial_1 x_S{}^j, \partial_2 x_S{}^k \partial_2 x_S{}^j\right)$

$\partial_1 x_S{}^j\left(\partial_1 x_S{}^j - \lambda \partial_2 x_S{}^j\right) = 0$

$\partial_2 x_S{}^k\left(\partial_1 x_S{}^k - \lambda \partial_2 x_S{}^k\right) = 0$

The coordinates x^j j=1,2,3, are Cartesian; hence the previous relations can be written in the form of dot products (Euclidean inner products) of the row vectors $\partial_1 x_S$=($\partial_1 x_S{}^1$,$\partial_1 x_S{}^2$,$\partial_1 x_S{}^3$) and $\partial_2 x_S$=($\partial_2 x_S{}^1$,$\partial_2 x_S{}^2$,$\partial_2 x_S{}^3$):

$\partial_1 x_S \cdot \left(\partial_1 x_S - \lambda \partial_2 x_S\right) = 0$

$\partial_2 x_S \cdot \left(\partial_1 x_S - \lambda \partial_2 x_S\right) = 0$

A necessary condition that the previous equations have a solution is:

$$\lambda = \frac{|\partial_1 x_S|}{|\partial_2 x_S|}$$

By substituting in any of them we result that:

$\partial_1 x_S \cdot \partial_2 x_S = |\partial_1 x_S||\partial_2 x_S|$

But according to the Cauchy-Schwarz inequality [5], this condition implies that the vectors $\partial_1 x_S$, $\partial_2 x_S$ are linearly dependent; which contradicts to our initial assumption.

We infer that the metric tensor is invertible and its determinant is different from zero at any point u=(u^1,u^2):

$\det g(u) \neq 0$

B) The length of a vector $\Delta x = e_\mu(u)\Delta u^\mu = e_\mu(u)\dot{u}^\mu(0)\Delta t \in T_u S$ is determined by the expression:

$|\Delta x| = \sqrt{\langle \Delta x, \Delta x\rangle} = \sqrt{g_{\mu\nu}(u)\Delta u^\mu \Delta u^\nu} = |\Delta t|\sqrt{g_{\mu\nu}(u)\dot{u}^\mu \dot{u}^\nu}$

Consequently, the infinitesimal length Δs on the curve $x(t) = x_S(u(t))$ lying on the surface S, is calculated by the relation:

$\Delta s^2 = \langle \Delta x(t), \Delta x(t) \rangle$

$\Delta x(t) = e_\mu(u(t)) \dot{u}^\mu(t) \Delta t$, $\Delta t \to 0$

$$\Delta s^2 = \langle \Delta x(t), \Delta x(t) \rangle = g_{\mu\nu}(u(t)) \Delta u^\mu \Delta u^\nu = g_{\mu\nu}(u(t)) \dot{u}^\mu(t) \dot{u}^\nu(t) (\Delta t)^2 \qquad (7.9)$$

C) For the case of a surface of revolution immersed in the 3-dimensional Euclidean space, the metric tensor on its tangent spaces is determined by the matrix:

$$[g_{\mu\nu}(u)] = \begin{pmatrix} (f(u^2))^2 & 0 \\ 0 & (f'(u^2))^2 + 1 \end{pmatrix} \qquad (7.10)$$

8. **1 and 2-forms on a surface**

We define the 1-forms on the surface $S: x^j = x_S{}^j(u^1, u^2)$ as the linear functions with domain any tangent space $T_u S$ of S and range in $\boldsymbol{R}$ (see paragraph 3).

All the properties of the forms referred in paragraphs 3 and 4 for the case of the 3-dimensional Euclidean space, also apply for the forms defined on the tangent planes of a surface. We repeat them here, in brief:

a) Let $\psi_{(u)} : T_u S \to \boldsymbol{R}$ be a 1-form on S. The action of $\psi_{(u)}$ on any vector $\xi = e_\mu(u) \xi^\mu \in T_u S$ returns the real number:

$\psi_{(u)}(\xi) = \psi_{(u)}(e_\nu(u) \xi^\nu) = \psi_{(u)}(e_\nu(u)) \xi^\nu \underset{def}{=} \psi_\nu(u) \xi^\nu$

The real functions $\psi_\nu(u)$ $\nu = 1, 2$ determine the 1-form $\psi_{(u)}$ at each tangent plane $T_u S$ of the surface.

We define the **vector field**:

$\xi_{(\psi)}(u) \underset{def}{=} e_\mu(u) \xi^\mu_{(\psi)}(u)$

With coordinates (see paragraph 3):

$\xi^\mu_{(\psi)}(u) = \psi_\lambda(u) g^{\lambda\mu}(u)$

The matrix $[g^{\mu\lambda}(u)]$ is the inverse of the metric tensor:

$[g^{\mu\lambda}(u)][g_{\lambda\nu}(u)] = [\delta^\mu_\nu]$

We can easily verify that the analytic expression the 1-form $\psi_{(u)}$ can be written as an inner product:

$\langle \xi_{(\psi)}(u), \xi \rangle = \langle e_\mu(u), e_\nu(u) \rangle \xi^\mu_{(\psi)}(u) \xi^\nu = g_{\mu\nu}(u) g^{\mu\lambda}(u) \psi_\lambda(u) \xi^\nu =$

$= g_{\nu\mu}(u) g^{\mu\lambda}(u) \psi_\lambda(u) \xi^\nu = \delta^\lambda_\nu \psi_\lambda(u) \xi^\nu = \psi_\nu(u) \xi^\nu = \psi_{(u)}(\xi)$

b) We define the basic 1-forms:

$\omega_\mu(\xi) \underset{def}{=} \langle e_\mu(u), \xi \rangle = \langle e_\mu(u), e_\nu(u) \xi^\nu \rangle = g_{\mu\nu}(u) \xi^\nu$

$\omega^\mu(\xi) = \omega^\mu(e_\nu \xi^\nu) = \xi^\mu$

It holds that:

$\omega_\mu = g_{\mu\nu} \omega^\nu$

Furthermore, any 1-form is possible to be expressed as a linear combination of the basic 1-forms:

$\psi(\xi) = \langle \xi_{(\psi)}, \xi \rangle = \xi^{\mu}_{(\psi)} \langle e_{\mu}, \xi \rangle = \xi^{\mu}_{(\psi)} \omega_{\mu}(\xi)$

Hence:

$\psi = \xi^{\mu}_{(\psi)} \omega_{\mu} = \xi^{\mu}_{(\psi)} g_{\mu\nu} \omega^{\nu}$

c) Consider a real function defined on the surface *S*, which is differentiable at least up to the second order with respect to each variable:

$F(u^1, u^2) = \tilde{F}\left(x^1{}_S(u), x^2{}_S(u), x^3{}_S(u)\right),\ u = (u^1, u^2)$

We define the **directional differential** of $F(u^1,u^2)$ at *u*, along the direction of the tangent vector $\Delta x = e_a(u)\Delta u^a \in T_u S$ by the relation:

$$d_{\Delta x} F(u) \underset{def}{=} \lim_{\tau \to 0} \frac{F(u + \tau \Delta u) - F(u)}{\tau} = \partial_a F(u) \Delta u^a \tag{8.1}$$

Given that $\Delta x = e_a(u)\Delta u^a \in T_u S$ there is a curve $u_t = c(t)$ of the parameters' space with the properties:

$c(0) = u = (u^1, u^2)$

$\dot{c}(0)\Delta t = \Delta u = (\Delta u^1, \Delta u^2),\ \Delta t \in \boldsymbol{R}$

Hence, relation 8.1 is written:

$$d_{\Delta x} F(u) \underset{def}{=} \lim_{\tau \to 0} \frac{F(u + \tau \Delta u) - F(u)}{\tau} = \partial_a F(u) \Delta u^a = \partial_a F(u) \dot{c}^a(0) \Delta t$$

The **directional derivative** of $F(u^1,u^2)$ at *u* is given by the equation:

$$\frac{d_{\Delta x} F(u)}{dt} = \partial_a F(u) \dot{c}^a(0)$$

By using the definition of the basic 1-forms ω^a we come to the conclusion that the directional differential of *F* is a 1-form defined on the tangent planes $T_u S$ given by the expression:

$$d_{\Delta x} F(u) = \partial_a F(u) \omega^a(\Delta x) \tag{8.2}$$

Consider the 1-form:

$\psi_{(u)} = \psi_a(u) \omega^a$

We say that the 1-form $\psi_{(u)}$ is an **exact form** if only we can find a real function $F(u^1,u^2)$ such that for any $u \in B$, $\Delta x \in T_u S$ it is true that:

$\psi_{(u)}(\Delta x) = d_{\Delta x} F(u)$

According to 8.2:

$\psi_{(u)} = dF(u) = \partial_a F(u) \omega^a$

Wedge product of two 1-forms: 2-forms on a surface

In accordance with what we have said in paragraph 4 about the 2-forms in a 3-dimensional Euclidean space, we come to a similar definition of the 2-forms on a surface: any 2-form on a surface is expressed as the wedge product of two 1-forms. Let $\omega_{(1)}$, $\omega_{(2)}$ be two 1-forms on *S*; their wedge product is determined by the equation:

$$(\omega_{(1)} \wedge \omega_{(2)})(\Delta_{(1)}x, \Delta_{(2)}x) \underset{def}{=} \omega_{(1)}(\Delta_{(1)}x)\omega_{(2)}(\Delta_{(2)}x) - \omega_{(1)}(\Delta_{(2)}x)\omega_{(2)}(\Delta_{(1)}x) \tag{8.3a}$$

$\Delta_{(1)}x, \Delta_{(2)}x \in T_u S$

The wedge product of two 1-forms $\omega_{(1)}$, $\omega_{(2)}$ -and consequently any 2-form- is a bilinear antisymmetric map of $T_uS \otimes T_uS$ into the set $\boldsymbol{R}$ of the real numbers.

Any 1-form is written as a linear combination of the basic 1-forms ω^μ (see the previous section of the present paragraph 8). Assume that:

$\omega_{(1)} = f_{(1)\mu}(u)\,\omega^\mu$

$\omega_{(2)} = f_{(2)\nu}(u)\,\omega^\nu$

The subsequent identities arise:

$$\omega_{(1)} \wedge \omega_{(2)} = \left(f_{(1)\mu}(u)\,\omega^\mu\right) \wedge \left(f_{(2)\nu}(u)\,\omega^\nu\right) = f_{(1)\mu}(u)\,f_{(2)\nu}(u)\,\omega^\mu \wedge \omega^\nu \qquad (8.3b)$$

We conclude that any 2-form σ can be written as a linear combination of the wedge product $\omega^\mu \wedge \omega^\nu$ of the basic 1-forms:

$$\sigma = f_{\mu\nu}(u)\,\omega^\mu \wedge \omega^\nu \qquad (8.4a)$$

The symbols $f_{\mu\nu}(u)$ stand for real functions defined on the parameters' space; as usually, the Greek indices run the values 1 and 2.

According to 8.3a, the non-identically zero wedge products in 8.4a are the next two:

$\omega^1 \wedge \omega^2$, $\omega^2 \wedge \omega^1 = -\omega^1 \wedge \omega^2$

We conclude that every 2-form defined on a surface *S* is determined by the analytic expression:

$$\sigma = f(u)\,\omega^1 \wedge \omega^2 \qquad (8.4b)$$

The map $f(u)$ is some real function of *B* into $\boldsymbol{R}$.

Integration of a 1-form along a curve lying on a surface - Exterior derivative of a 1-form - Another case of the Stokes' theorem

The concept of the 2-forms is intimately related with the 1-forms. By following the same reasoning path as in paragraphs 3 and 4, we show that this relation becomes clearer by analyzing the result of the integration of a 1-form along the boundary of an infinitesimal parallelogram of the parameters' space. This process will lead us to a formulation of the Stokes' theorem for the case of 1 and 2-forms determined on surfaces.

Assume a curve $u_{(1)} : u^\mu = {u_{(1)}}^\mu(t)$, $t \in I \subseteq \boldsymbol{R}$ in the domain *B* of the surface:

$S : x^j = {x_S}^j\left(u^1, u^2\right)$

The image of $u_{(1)}$ on the surface is the curve:

$x_S \circ u_{(1)} : x = x_S\left(u_{(1)}(t)\right) = x_S\left({u_{(1)}}^1(t), {u_{(1)}}^2(t)\right)$

At every point of the curve $x_S \circ u_{(1)}$ a unique tangent plane $T_{u_{(1)}(t)}S$ of *S* is determined. Assume a certain 1-form $\omega_p = p_\alpha\left(u_{(1)}(t)\right)\omega^\alpha$ defined on the tangent spaces $T_{u_{(1)}(t)}S$ for every *t* in the interval *I*. The tangent vectors of the curve $x_S \circ u_{(1)}$ as *t* runs *I*, are given by the relationship:

$$\Delta_{(1)}x(t) = e_\nu\left(u_{(1)}(t)\right)\dot{u}_{(1)}{}^\nu(t)\Delta t = e_\nu\left(u_{(1)}(t)\right)\Delta u_{(1)}{}^\nu(t) \in T_{u_{(1)}(t)}S \qquad (8.5)$$

$\Delta u_{(1)}{}^\nu(t) = \dot{u}_{(1)}{}^\nu(t)\Delta t \quad \Delta t \in \boldsymbol{R}$

Relation 8.5 defines a vector field determined along the curve $x_S \circ u_{(1)}$ of *S*. The value returned by the 1-form ω_p at each vector $\Delta_{(1)}x(t)$ of the field is:

$$\omega_p\left(\Delta_{(1)}x(t)\right)=p_\mu\left(u_{(1)}(t)\right)\dot{u}_{(1)}{}^\mu(t)\Delta t=p_\mu\left(u_{(1)}(t)\right)\Delta_{(1)}u^\mu(t) \tag{8.6}$$

We define the integral of the 1-form ω_p on the curve $x_S\circ u_{(1)}$ by the expression:

$$\int\limits_{x_S\circ u_{(1)}}\omega_p=\int\limits_{x_S\circ u_{(1)}}p_a(u)\omega^a\underset{def}{=}\int\limits_{u_{(1)}}p_\mu\left(u_{(1)}(t)\right)\dot{u}_{(1)}{}^\mu(t)dt \tag{8.7}$$

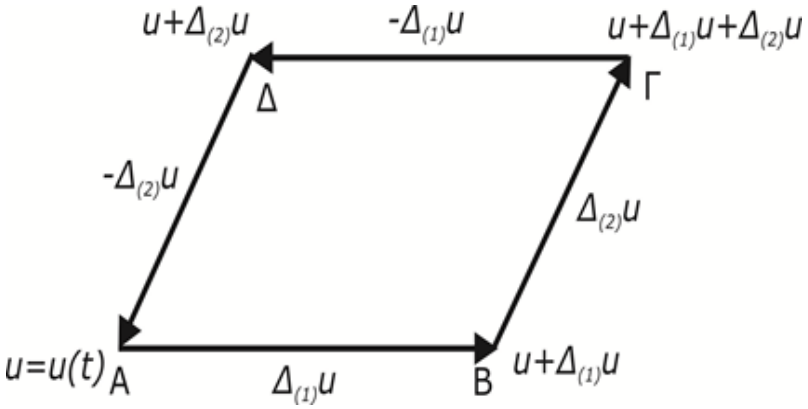

Figure 8.1: The boundary of an infinitesimal parallelogram lying on the domain *B* of *S*. Its image $x_S\circ\partial\Pi_u[\Delta_{(1)}u,\Delta_{(2)}u]$ is a simple closed curve of *S*.

Let us assume that the curve $u_{(1)}:u^\mu=u_{(1)}{}^\mu(t)$ is the boundary $\partial\Pi_u[\Delta_{(1)}u,\Delta_{(2)}u]$ of an **infinitesimal parallelogram** $\Pi_u[\Delta_{(1)}u,\Delta_{(2)}u]$ of the parameters' space $B\subseteq\boldsymbol{R}^2$ (figure 8.1). We presume that the image-curve $x_S\circ\partial\Pi_u[\Delta_{(1)}u,\Delta_{(2)}u]$ is a simple and closed infinitesimal curve on *S*. We integrate ω_p along $\partial\Pi_u[\Delta_{(1)}u,\Delta_{(2)}u]$ according to 8.7; we apply the mean value theorem and expand the functions $p_a\left(u+\Delta_{(1)}u\right)$, $p_a\left(u+\Delta_{(2)}u\right)$ in Taylor series keeping terms up to the first order with respect to the quantities $\Delta_{(1)}u^\alpha$, $\Delta_{(2)}u^\beta$ (see paragraph 4).

$$\oint\limits_{x_S\circ\partial\Pi_u[\Delta_{(1)}u,\Delta_{(2)}u]}\omega_p=\oint\limits_{x_S\circ\partial\Pi_u[\Delta_{(1)}u,\Delta_{(2)}u]}p_a(u)\omega^a=$$

$$=p_a(u)\Delta_{(1)}u^a+p_a\left(u+\Delta_{(1)}u\right)\Delta_{(2)}u^a-p_a\left(u+\Delta_{(2)}u\right)\Delta_{(1)}u^a-p_a(u)\Delta_{(2)}u^a=$$

$$=\partial_\beta p_a(u)\left(\Delta_{(1)}u^\beta\Delta_{(2)}u^a-\Delta_{(1)}u^a\Delta_{(2)}u^\beta\right)=\partial_\beta p_a(u)\omega^\beta\wedge\omega^a\left(\Delta_{(1)}x,\Delta_{(2)}x\right)$$

$$\oint\limits_{x_S\circ\partial\Pi_u[\Delta_{(1)}u,\Delta_{(2)}u]}\omega_p=\oint\limits_{x_S\circ\partial\Pi_u[\Delta_{(1)}u,\Delta_{(2)}u]}p_a(u)\omega^a=\partial_\beta p_a(u)\omega^\beta\wedge\omega^a\left(\Delta_{(1)}x,\Delta_{(2)}x\right) \tag{8.8}$$

From 8.8 we come to the definition of the **exterior derivative $d\omega_p$** of the 1-form $\omega_p=p_a(u)\omega^a$ as the 2-form given by the expression:

$$d\omega_p=\partial_\beta p_a(u)\omega^\beta\wedge\omega^a \tag{8.9}$$

The combination of 8.8 and 8.9 leads to the equation:

$$d\omega_p\left(\Delta_{(1)}x,\Delta_{(2)}x\right)=\partial_\beta p_a(u)\omega^\beta\wedge\omega^a\left(\Delta_{(1)}x,\Delta_{(2)}x\right)=\oint\limits_{x_S\circ\partial\Pi_u[\Delta_{(1)}u,\Delta_{(2)}u]}\omega_p \tag{8.10a}$$

Let us now consider a compact and connected subset R_c of *B*. The boundary of R_c is the closed curve $c=\partial R_c$ (see paragraph 4: the exterior derivative of a 1-form). The set R_c is possible to be approximated by a collection of infinitesimal parallelograms (figure 4.2). For

any 1-form ω_p we apply a procedure similar to the one of the paragraph 4 and we obtain the relationship:

$$\oint_{\partial R_c} \omega_p = \int_{R_c} d\omega_p \tag{8.10b}$$

Equation 8.10b is a formulation of the Stokes' theorem for the case of 1 and 2-forms defined on surfaces.

Proposition 8.1
The integral of an exact 1-form ω_p along any closed curve *c* of *B* equals to zero. Inversely: if the integral of a 1-form ω_p along any closed curve *c* of *B* equals to zero, then ω_p is an exact form.
Hint: The steps to the proof are similar to the ones of the Proposition 4.1

Corollary: The necessary and sufficient condition for $\omega_p = p_a(u)\omega^a$ to be exact is

$$\partial_1 p_2(u) = \partial_2 p_1(u) \tag{8.11}$$

9. **Parameter transformations - Invariant 2-forms - The area-form on a surface**
In this paragraph we examine how do the main geometric concepts defined up to now on a simple surface *S*, vary under a diffeomorphic transformation of the parameters. We formulate the transformation-laws of the basis-elements, the metric tensor and the 1 and 2-forms on the tangent planes of the surface. The area-form is defined as the 2-form that is invariant under any parameters' transformation.

We consider the general group of the diffeomorphisms (see paragraph 5) for the parameters' space of a surface *S*. Recall that each parameter transformation $u^\mu = u^\mu(\bar{u})$ is one-to-one and onto, has derivatives at least up to the second order and its inverse transformation $\bar{u}^\mu = \bar{u}^\mu(u)$ is differentiable up to the second order too.

The Jacobian matrix of the transformation $u^\mu = u^\mu(\bar{u})$ is:

$$J = \left[J^\alpha_\beta\right] = \begin{pmatrix} \frac{\partial u^1}{\partial \bar{u}^1} & \frac{\partial u^1}{\partial \bar{u}^2} \\ \frac{\partial u^2}{\partial \bar{u}^1} & \frac{\partial u^2}{\partial \bar{u}^2} \end{pmatrix} \underset{def}{=} \begin{pmatrix} \bar{\partial}_1 u^1 & \bar{\partial}_2 u^1 \\ \bar{\partial}_1 u^2 & \bar{\partial}_2 u^2 \end{pmatrix}$$

The Jacobian matrix of the inverse transformation $\bar{u}^\mu = \bar{u}^\mu(u)$ is:

$$\bar{J} = \left[\bar{J}^\alpha_\beta\right] = \begin{pmatrix} \frac{\partial \bar{u}^1}{\partial u^1} & \frac{\partial \bar{u}^1}{\partial u^2} \\ \frac{\partial \bar{u}^2}{\partial u^1} & \frac{\partial \bar{u}^2}{\partial u^2} \end{pmatrix} = \begin{pmatrix} \partial_1 \bar{u}^1 & \partial_2 \bar{u}^1 \\ \partial_1 \bar{u}^2 & \partial_2 \bar{u}^2 \end{pmatrix}$$

The matrix $\bar{J}$ is the inverse of the matrix *J*:

$$\frac{\partial u^\mu}{\partial u^\nu} = \frac{\partial u^\mu}{\partial \bar{u}^\kappa}\frac{\partial \bar{u}^\kappa}{\partial u^\nu} \quad \frac{\partial u^\mu}{\partial \bar{u}^\kappa}\frac{\partial \bar{u}^\kappa}{\partial u^\nu} = \delta^\mu_\nu$$

Hence:

$$\bar{J} = J^{-1}$$

Transformation of the tangent planes' basis-elements under a parameter transformation

With respect to the u-parameters system, the Cartesian coordinates of the points of *S* are determined by the functions:

$x^j = x_S{}^j(u)$, $u = \left(u^1, u^2\right)$, $x_S(u) = \left(x_S{}^1(u), x_S{}^2(u), x_S{}^3(u)\right)$

Under the parameter-transformation $u^\mu = u^\mu\left(\bar{u}\right)$ we can write (paragraph 6):

$$x_S(u) = x_S\left(u\left(\bar{u}\right)\right) \underset{def}{=} \bar{x}_S\left(\bar{u}\right) \tag{9.1}$$

In the new set of parameters the basis-elements of each tangent plane of *S* are calculated by 7.5 and 9.1:

$$e_\mu(u) = \frac{\partial x(u)}{\partial u^\mu} = \frac{\partial \bar{x}(\bar{u})}{\partial u^\mu} = \frac{\partial \bar{x}(\bar{u})}{\partial \bar{u}^\nu}\frac{\partial \bar{u}^\nu}{\partial u^\mu} = \bar{e}_\nu(\bar{u})\bar{J}_\mu^\nu \tag{9.2a}$$

$$\bar{e}_\mu(\bar{u}) = e_\nu(u)J_\mu^\nu \tag{9.2b}$$

Under any parameter-transformation any vector $\Delta x = e_\mu \Delta u^\mu \in T_u S$ remains invariant, although the basis-elements of $T_u S$ are being changed. This implies that the coordinates of Δx with respect to the new parameters have been changed too. We write:

$\Delta x = \Delta \bar{x}$

$e_\mu \Delta u^\mu = \bar{e}_\nu \Delta \bar{u}^\nu = e_\mu J_\nu^\mu \Delta \bar{u}^\nu$

$$\Delta u^\mu = J_\nu^\mu \Delta \bar{u}^\nu \tag{9.3a}$$

$$\Delta \bar{u}^\mu = \bar{J}_\nu^\mu \Delta u^\nu \tag{9.3b}$$

Transformation of the metric tensor

The metric tensor of a tangent plane $T_u S$ transforms according to the relations:

$$g_{\mu\nu} = \langle e_\mu, e_\nu \rangle = \langle \bar{e}_\kappa, \bar{e}_\lambda \rangle \bar{J}_\mu^\kappa \bar{J}_\nu^\lambda = \bar{g}_{\kappa\lambda}\bar{J}_\mu^\kappa \bar{J}_\nu^\lambda \tag{9.4a}$$

$$\bar{g}_{\kappa\lambda} = g_{\mu\nu} J_\kappa^\mu J_\lambda^\nu \tag{9.4b}$$

In matrix form:

$$\bar{g} = J^T g J \tag{9.4c}$$

$$g = \bar{J}^T \bar{g} \bar{J} \tag{9.4d}$$

The determinants of a matrix and its transpose are equal:

$\det J = \det J^T$

Hence, from 9.4 we imply the subsequent relationships:

$\det \bar{g} = \det g \cdot (\det J)^2$

$$\det J = \sqrt{\frac{\det \bar{g}}{\det g}} \tag{9.5}$$

(Notice that $\det g(u) \neq 0$ for any *u*, as we have pointed out in paragraph 7: the metric tensor of the tangent planes of the surface *S*)

Another consequence of the equations 9.4 is that the infinitesimal length Δs along any curve $x(t) = x_S(u(t))$ lying on the surface *S* does not change under any parameter-transformation:

$\Delta s^2 = g_{\mu\nu}\Delta u^\mu \Delta u^\nu = \bar{g}_{\kappa\lambda}\bar{J}_\mu^\kappa \bar{J}_\nu^\lambda J_\rho^\mu \Delta \bar{u}^\rho J_\sigma^\nu \Delta \bar{u}^\sigma = \bar{g}_{\kappa\lambda}\Delta \bar{u}^\kappa \Delta \bar{u}^\lambda = \Delta \bar{s}^2$

$\Delta u^\mu = \dot{u}^\mu(t)\Delta t$, $\Delta t \to 0$

Remark: Any set of quantities $T^{\kappa\lambda\ldots}_{\mu\nu\ldots}(u)$ defined on *S*, is called a **tensor field** if only under a parameter-transformation $u^\mu = u^\mu(\bar{u})$ transform according to the relation:

$\bar{T}^{\kappa\lambda\ldots}_{\mu\nu\ldots}(\bar{u}) = T^{\kappa'\lambda'\ldots}_{\mu'\nu'\ldots}(u)\bar{J}^{\kappa}_{\kappa'}(u)\bar{J}^{\lambda}_{\lambda'}(u)\ldots J^{\mu'}_{\mu}(u)J^{\nu'}_{\nu}(u)\ldots$

The metric tensor is a tensor field.

Transformation of the 1-forms

To find out how the 1-forms transform under the parameters-transformation $u^\mu = u^\mu(\bar{u})$ we follow the procedure we applied in paragraph 6, where we derived the transformation-rule of the 1-forms on a Euclidean space under a coordinate-transformation.

Consider the 1-form $\omega_p = p_\mu(u)\omega^\mu$ expressed in the u-parameters. The same 1-form in the $\bar{u}$-parameters gets the analytic expression:

$\bar{\omega}_p = P_\nu(\bar{u})\bar{\omega}^\nu$

As usually, for any $\Delta x = e_\mu(u)\Delta u^\mu = \bar{e}_\nu(\bar{u})\Delta\bar{u}^\nu$ we have:

$\omega_p(\Delta x) = \bar{\omega}_p(\Delta\bar{x})$

$p_\nu(u)\omega^\mu(\Delta x) = \bar{p}_\mu(\bar{u})\bar{\omega}^\mu(\Delta\bar{x})$

$p_\nu(u)\Delta u^\nu = \bar{p}_\mu(\bar{u})\Delta\bar{u}^\mu$

$p_\nu(u)J^\nu_\mu\Delta\bar{u}^\mu = \bar{p}_\mu(\bar{u})\Delta\bar{u}^\mu$

$$\bar{p}_\mu(\bar{u}) = p_\nu(u)J^\nu_\mu \tag{9.6}$$

For the basic 1-forms ω^μ we are getting:

$\omega^\mu(\Delta x) = \Delta u^\mu = J^\mu_\nu\Delta\bar{u}^\nu = J^\mu_\nu\bar{\omega}^\nu(\Delta\bar{x})$

Hence:

$$\omega^\mu = J^\mu_\nu\bar{\omega}^\nu \tag{9.7}$$

Transformation of the 2-forms - The area element on a surface

A 2-form σ defined on a surface *S* is expressed by the relation 8.4b: $\sigma = f(u)\omega^1\wedge\omega^2$

How does σ transform under the parameter-transformation?

Let $\bar{\sigma} = \bar{f}(\bar{u})\bar{\omega}^1\wedge\bar{\omega}^2$ be the analytic expression of σ in the $\bar{u}$-parameters.

For any $\Delta_{(1)}x, \Delta_{(2)}x \in T_uS$ we have:

$\sigma(\Delta_{(1)}x, \Delta_{(2)}x) = \bar{\sigma}(\Delta_{(1)}\bar{x}, \Delta_{(2)}\bar{x})$

$f(u)\omega^1\wedge\omega^2(\Delta_{(1)}x, \Delta_{(2)}x) = \bar{f}(\bar{u})\bar{\omega}^1\wedge\bar{\omega}^2(\Delta_{(1)}\bar{x}, \Delta_{(2)}\bar{x})$

$f(u)J^1_\mu J^2_\nu\bar{\omega}^\mu\wedge\bar{\omega}^\nu(\Delta_{(1)}\bar{x}, \Delta_{(2)}\bar{x}) = \bar{f}(\bar{u})\bar{\omega}^1\wedge\bar{\omega}^2(\Delta_{(1)}\bar{x}, \Delta_{(2)}\bar{x})$

$f(u)(J^1_1J^2_2 - J^1_2J^2_1)\bar{\omega}^1\wedge\bar{\omega}^2(\Delta_{(1)}\bar{x}, \Delta_{(2)}\bar{x}) = \bar{f}(\bar{u})\bar{\omega}^1\wedge\bar{\omega}^2(\Delta_{(1)}\bar{x}, \Delta_{(2)}\bar{x})$

$$\bar{f}(\bar{u}) = f(u)\det J \tag{9.8}$$

From 9.5 and 9.8 we imply that:

$$\frac{\bar{f}(\bar{u})}{\sqrt{|\det\bar{g}(\bar{u})|}} = \frac{f(u)}{\sqrt{|\det g(u)|}} \tag{9.9}$$

The invariance of σ under the transformation $u^\mu = u^\mu(\bar{u})$ is ensured if only the following condition is satisfied (see paragraph 6):

$$\bar{f}(\bar{u}) = f(\bar{u}) \tag{9.10a}$$

By 9.9 and 9.10a we imply that a sufficient and necessary condition for σ to be invariant under the parameter transformation $u^{\mu} = u^{\mu}(\bar{u})$ is:

$$\frac{f(u)}{\sqrt{|\det g(u)|}} = \frac{f(\bar{u})}{\sqrt{|\det \bar{g}(\bar{u})|}} \tag{9.10b}$$

That means that the value of the quotient $f(u)\Big/\sqrt{|\det g(u)|}$ should be independent of the choice of the parameters; hence should be equal to a constant number. We conclude that the 2-forms which are invariant under the transformations $u^{\mu} = u^{\mu}(\bar{u})$ have the analytic expression:

$\sigma_{inv.} = \lambda\sqrt{\det g(u)}\,\omega^1 \wedge \omega^2$, $\lambda = \text{constant}$

We choose the System of Units so that $\lambda = 1$ and we define the area-form on the surface *S* to be the 2-form:

$$\sigma \underset{def}{=} \sqrt{\det g(u)}\,\omega^1 \wedge \omega^2 = \sqrt{\det \bar{g}(\bar{u})}\,\bar{\omega}^1 \wedge \bar{\omega}^2 \tag{9.11}$$

The infinitesimal area *da* determined by the infinitesimal vectors $\Delta_{(1)}x, \Delta_{(2)}x \in T_u S$ is calculated by applying 9.11:

$$\begin{aligned} da &= \sigma\left(\Delta_{(1)}x, \Delta_{(2)}x\right) = \sqrt{\det g(u)}\,\omega^1 \wedge \omega^2\left(\Delta_{(1)}x, \Delta_{(2)}x\right) = \\ &= \sqrt{\det g(u)}\left(\Delta_{(1)}u^1\Delta_{(2)}u^2 - \Delta_{(2)}u^1\Delta_{(1)}u^2\right) \end{aligned} \tag{9.12}$$

Example 9A

Calculation of the area of a surface of revolution and of the volume enclosed by it
Application for the case of a sphere

In the present Example we are working out two applications based on the results of paragraph 9: a) a general expression for the calculation of the surface-area for any surface of revolution is achieved, b) we calculate the volume enclosed by a surface of revolution in the 3-dimensional Euclidean space by integrated a certain 2-form on it; the analytic expression of this 2-form is obtained by applying Stokes' theorem and looking for a 2-form with exterior derivative identical to the volume-area of the 3-dimensional Euclidean space.

a) Calculation of the area of a surface of revolution

Consider the surface of revolution in the 3-dimensional Euclidean space:

$$S_f : \begin{cases} x^1 = f(u^2)\cos u^1 \\ x^2 = f(u^2)\sin u^1 \\ x^3 = u^2 \end{cases} \tag{E9A.1}$$

We assume that:

$f\left(u^2\right) \geq 0$, $u \in B$

$f'(u^2) \underset{def}{=} \dfrac{df(u^2)}{du^2}$

The basis vectors of the tangent planes $T_u S_f$ are

$e_1 = -\boldsymbol{x}_1 f(u^2)\sin u^1 + \boldsymbol{x}_2 f(u^2)\cos u^1$

$e_2 = \boldsymbol{x}_1 f'(u^2)\cos u^1 + \boldsymbol{x}_2 f'(u^2)\sin u^1 + \boldsymbol{x}_3$

The matrix-elements of the metric tensor are:

$$g_{11} = \langle e_1, e_1 \rangle = \left(f\left(u^2\right)\right)^2$$

$$g_{12} = g_{21} = 0$$

$$g_{22} = 1 + \left(f'\left(u^2\right)\right)^2$$

$$g(u) \underset{def}{=} \left[g_{\mu\nu}(u) \right] = \begin{pmatrix} \left(f\left(u^2\right)\right)^2 & 0 \\ 0 & 1 + \left(f'\left(u^2\right)\right)^2 \end{pmatrix}$$

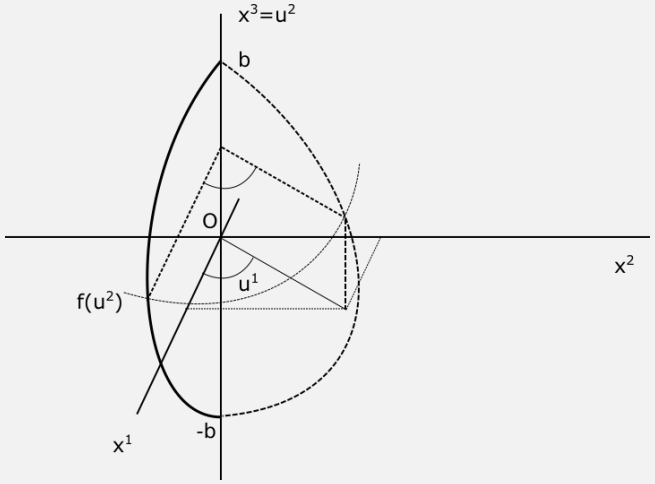

Figure 9.1: A surface of revolution S_f.

The area form on S_f is:

$$\sigma_f = \sqrt{\det g(u)}\, \omega^1 \wedge \omega^2 = f\left(u^2\right) \sqrt{1 + \left(f'\left(u^2\right)\right)^2}\, \omega^1 \wedge \omega^2$$

The infinitesimal parallelogram $\Pi_x[\Delta_{(1)}x, \Delta_{(2)}x]$ has vertex at x and sides:

$\Delta_{(1)}x = e_\mu \Delta_{(1)}u^\mu$, $\Delta_{(2)}x = e_\mu \Delta_{(2)}u^\mu$

The elementary area of $\Pi_x[\Delta_{(1)}x, \Delta_{(2)}x]$ is given by the relationship:

$$\Delta a = \sigma_f \left(\Delta_{(1)}x, \Delta_{(2)}x \right) = f\sqrt{f'^2 + 1} \left(\Delta_{(1)}u^1 \Delta_{(2)}u^2 - \Delta_{(2)}u^1 \Delta_{(1)}u^2 \right) \qquad \text{(E9A.2a)}$$

We assume that:

$\Delta_{(1)}x = e_1 \Delta u^1$, $\Delta_{(2)}x = e_2 \Delta u^2$

Then, E9A.2a takes the form:

$$\Delta a = f\sqrt{f'^2 + 1}\, \Delta u^1 \Delta u^2 \qquad \text{(E9A.2b)}$$

Hence, the area of the surface seen in figure 9.1, if it exists, is calculated by the formula:

$$\text{area}\left(S_f\right) = \int_{S_f} du^1 du^2 f\left(u^2\right) \sqrt{1 + \left(f'\left(u^2\right)\right)^2} = 2\pi \int_{-b}^{b} du^2 f\left(u^2\right) \sqrt{1 + \left(f'\left(u^2\right)\right)^2} \qquad \text{(E9A.3)}$$

For the case that S_f is a sphere of radius b, the function $f(u^2)$ is given by the expression:

$$f\left(u^2\right) = \sqrt{b^2 - (u^2)^2}$$

The derivative of $f(u^2)$ is:

$$f'\left(u^2\right) = -u^2 \left(b^2 - (u^2)^2 \right)^{-1/2}$$

We substitute in 9A.3 and we obtain the expected relation:

$$\text{area}(S_f) = 2\pi \int_{-b}^{b} du^2 \sqrt{b^2 - (u^2)^2} \left(1 + \frac{(u^2)^2}{b^2 - (u^2)^2}\right)^{1/2} = 2\pi \int_{-b}^{b} du^2 b = 4\pi b^2$$

b) Calculation of the volume enclosed by a surface of revolution *S* by integrating an appropriate 2-form on *S*

Assume a 2-form of the 3-dimensional Euclidean space $\boldsymbol{R}_0^3$ (see paragraph 4):

$$\omega_p = p_{jk}(x) dx^j \wedge dx^k \qquad \text{(E9A.4)}$$

Let *R* be any region of $\boldsymbol{R}_0^3$ whose boundary $S=\partial R$ is a surface of revolution. Then by applying the Stokes' theorem (relation 6.11) we obtain the identity:

$$\oint_{\partial R} p_{jk}(x) dx^j \wedge dx^k = \int_R \partial_i p_{jk}(x) dx^i \wedge dx^j \wedge dx^k \qquad \text{(E9A.5)}$$

The volume of *R* is given by the integral (paragraph 6):

$$\text{vol}(R) = \int_R dx^1 \wedge dx^2 \wedge dx^3 \qquad \text{(E9A.6)}$$

Is it possible to choose the form ω_p *so that the right part of E9A.5 to be identical to the volume of the region R? That is:*

$$\int_R dx^1 \wedge dx^2 \wedge dx^3 = \int_R \partial_i p_{jk}(x) dx^i \wedge dx^j \wedge dx^k \qquad \text{(E9A.7)}$$

We can easily verify the identity:

$$dx^1 \wedge dx^2 \wedge dx^3 = \frac{1}{6} \varepsilon_{ijk} dx^i \wedge dx^j \wedge dx^k$$

Then, one appropriate choice of $p_{jk}(x)$ should be the solutions of the equations:

$$\partial_i p_{jk}(x) = \frac{1}{6} \varepsilon_{ijk} \qquad \text{(E9A.8)}$$

A legitimate solution of E9A.8 is:

$$p_{11} = p_{22} = p_{33} = 0$$

$$p_{12} = -p_{21} = \frac{x^3}{6}$$

$$p_{23} = -p_{32} = \frac{x^1}{6}$$

$$p_{31} = -p_{13} = \frac{x^2}{6}$$

We result that a 2-form satisfying E9A.7 is the following:

$$\omega = \frac{1}{3}\left(x^3 dx^1 \wedge dx^2 + x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1\right) \qquad \text{(E9A.9)}$$

Hence, the volume enclosed by the surface $S=\partial R$ is possible to be calculated by the expression:

$$\text{vol}(R) = \frac{1}{3} \oint_{S=\partial R} \left(x^3 dx^1 \wedge dx^2 + x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1\right) \qquad \text{(E9A.10)}$$

Let us apply E9A.10 to calculate the volume enclosed by a surface of revolution *S*, determined by the analytic expression $x^j = x_S^j(u^1, u^2)$.

At the right hand side of E9A.10, the 2-forms $dx^i \wedge dx^j$ are defined on the tangent spaces $T_{x_S(u)}\boldsymbol{R}_0^3$ but act on the vectors $\Delta_{(1)}x = e_1 \Delta u^1$, $\Delta_{(2)}x = e_2 \Delta u^2$ of the tangent planes $T_u S$ of *S*. Each tangent plane $T_u S$ is a subspace of the corresponding tangent space $T_{x_S(u)}\boldsymbol{R}_0^3$ of the Euclidean space. Hence, in order to reform the integral at E9A.10 to a double integral with

respect to the "free" parameters u^1, u^2 we have to calculate the wedge-products as functions of u^1, u^2. We are getting:

$$dx^1 \wedge dx^2\left(\Delta_{(1)}x, \Delta_{(2)}x\right) = dx^1\left(\Delta_{(1)}x\right)dx^2\left(\Delta_{(2)}x\right) - dx^1\left(\Delta_{(2)}x\right)dx^2\left(\Delta_{(1)}x\right)$$

…

$$\Delta_{(1)}x = e_1\Delta u^1 = \boldsymbol{x}_j\,\frac{\partial x_S{}^j}{\partial u^1}\,\Delta u^1, \quad \Delta_{(2)}x = e_2\Delta u^2 = \boldsymbol{x}_k\,\frac{\partial x_S{}^k}{\partial u^2}\,\Delta u^2$$

$$dx^1\left(\Delta_{(1)}x\right) = \frac{\partial x_S{}^1}{\partial u^1}\Delta u^1, \quad dx^1\left(\Delta_{(2)}x\right) = \frac{\partial x_S{}^1}{\partial u^2}\Delta u^2$$

$$dx^2\left(\Delta_{(1)}x\right) = \frac{\partial x_S{}^2}{\partial u^1}\Delta u^1, \quad dx^2\left(\Delta_{(2)}x\right) = \frac{\partial x_S{}^2}{\partial u^2}\Delta u^2$$

$$dx^3\left(\Delta_{(1)}x\right) = \frac{\partial x_S{}^3}{\partial u^1}\Delta u^1, \quad dx^3\left(\Delta_{(2)}x\right) = \frac{\partial x_S{}^3}{\partial u^2}\Delta u^2$$

$$dx^1 \wedge dx^2\left(\Delta_{(1)}x, \Delta_{(2)}x\right) = \left(\frac{\partial x_S{}^1}{\partial u^1}\frac{\partial x_S{}^2}{\partial u^2} - \frac{\partial x_S{}^1}{\partial u^2}\frac{\partial x_S{}^2}{\partial u^1}\right)\Delta u^1\Delta u^2 = \frac{\partial\left(x_S{}^1, x_S{}^2\right)}{\partial\left(u^1, u^2\right)}\Delta u^1\Delta u^2 \quad \text{(E9A.11)}$$

$$\frac{\partial\left(x_S{}^1, x_S{}^2\right)}{\partial\left(u^1, u^2\right)} \underset{def}{=} \det\begin{pmatrix} \frac{\partial x_S{}^1}{\partial u^1} & \frac{\partial x_S{}^2}{\partial u^1} \\ \frac{\partial x_S{}^1}{\partial u^2} & \frac{\partial x_S{}^2}{\partial u^2} \end{pmatrix}$$

Similarly, we find that:

$$dx^2 \wedge dx^3\left(\Delta_{(1)}x, \Delta_{(2)}x\right) = \frac{\partial\left(x_S{}^2, x_S{}^3\right)}{\partial\left(u^1, u^2\right)}\Delta u^1\Delta u^2 \quad \text{(E9.A12)}$$

$$dx^3 \wedge dx^1\left(\Delta_{(1)}x, \Delta_{(2)}x\right) = \frac{\partial\left(x_S{}^3, x_S{}^1\right)}{\partial\left(u^1, u^2\right)}\Delta u^1\Delta u^2 \quad \text{(E9.A13)}$$

We substitute in E9A.10 and we result the following relationship which is very useful and applicable when the analytic expression of the surface is known:

$$\mathrm{vol}(R) = \frac{1}{3}\oint_{S=\partial R} du^1du^2\left(x_S{}^1\frac{\partial\left(x_S{}^2, x_S{}^3\right)}{\partial\left(u^1, u^2\right)} + x_S{}^2\frac{\partial\left(x_S{}^3, x_S{}^1\right)}{\partial\left(u^1, u^2\right)} + x_S{}^3\frac{\partial\left(x_S{}^1, x_S{}^2\right)}{\partial\left(u^1, u^2\right)}\right) \quad \text{(E9A.14)}$$

Remark: The analysis of the section b of the present example holds for any compact simply connected surface S [(2)]. The surfaces of revolution are a special case of this wider class of surfaces [(2)].

Application of E9A.14 for the case of the surface of revolution determined by the analytic expression E9A.1: Consider the surface of revolution S_f (relation E9A.1). We calculate the determinants appearing at the right hand side of E9A.14:

$$\frac{\partial\left(x_S{}^2, x_S{}^3\right)}{\partial\left(u^1, u^2\right)} = \det\begin{pmatrix} \partial_1x^2 & \partial_1x^3 \\ \partial_2x^2 & \partial_2x^3 \end{pmatrix} = \det\begin{pmatrix} f\cos u^1 & 0 \\ f'\sin u^1 & 1 \end{pmatrix} = f\cos u^1$$

$$\frac{\partial\left(x_S{}^3, x_S{}^1\right)}{\partial\left(u^1, u^2\right)} = \det\begin{pmatrix} \partial_1x^3 & \partial_1x^1 \\ \partial_2x^3 & \partial_2x^1 \end{pmatrix} = \det\begin{pmatrix} 0 & -f\sin u^1 \\ 1 & f'\cos u^1 \end{pmatrix} = f\sin u^1$$

$$\frac{\partial\left(x_S{}^1, x_S{}^2\right)}{\partial\left(u^1, u^2\right)} = \det\begin{pmatrix} \partial_1x^1 & \partial_1x^2 \\ \partial_2x^3 & \partial_2x^1 \end{pmatrix} = \det\begin{pmatrix} -f\sin u^1 & f\cos u^1 \\ f'\cos u^1 & f'\sin u^1 \end{pmatrix} = -ff'$$

We substitute in E9A.14 and we find that the volume enclosed by the surface of revolution is calculated by the expression:

$$\mathrm{vol}(R_f)=\frac{1}{3}\oint_{S_f} du^1du^2f\left(u^2\right)\left(f\left(u^2\right)-f'\left(u^2\right)u^2\right)=\frac{2\pi}{3}\int_{-b}^{b} du^2f\left(u^2\right)\left(f\left(u^2\right)-f'\left(u^2\right)u^2\right) \quad \text{(E9A.15)}$$

Application of 9A.15 for the case of a sphere: A sphere can be considered as a surface of revolution. The curve of the plane Ox^1x^3 which is to be revolved for creating a sphere with radius *b* is the semi-circle:

$f(u^2)=\left(b^2-(u^2)^2\right)^{1/2}$

We insert this function in E9A.15 and we perform the integration. We are getting the expected result:

$$\mathrm{vol}(Sphere)=\frac{4}{3}\pi b^3$$

Another expression for the volume enclosed by a surface of revolution S_f resulting from E9A.15: By performing integration by parts at the last integral of the expression E9A.15, we obtain:

$$\mathrm{vol}(R_f)=\frac{2\pi}{3}\int_{-b}^{b} du^2f^2-\frac{2\pi}{3}\left[f^2u^2\right]_{-b}^{b}+\frac{2\pi}{3}\int_{-b}^{b} du^2f^2+\frac{2\pi}{3}\int_{-b}^{b} du^2ff'u^2$$

We keep the restriction $f(b)=f(-b)=0$ (figure 9.1) and from the last equation we result that:

$$\mathrm{vol}(R_f)=2\pi\int_{-b}^{b} du^2f^2-\mathrm{vol}(R_f)$$

$$\mathrm{vol}(R_f)=\pi\int_{-b}^{b} du^2\left(f(u^2)\right)^2 \quad \text{(E9A.16)}$$

10. **The geometric surface**

Any point of a surface $S:x^j=x_S^j(u^1,u^2)$ which is immersed in a 3-dimensional Euclidean or pseudo-Euclidean space is determined by the values of the couple (u^1, u^2). In this paragraph we make an abstraction: we try to conceive a surface *S* freed from the underlying space, as an autonomous entity; we imagine that *S* is like a plane-space whose points are determined by the parameters u^1, u^2: $S \ni P_u \leftrightarrow u=\left(u^1,u^2\right)\in B\subseteq \boldsymbol{R}^2$

The tangent spaces T_uS of *S* are equipped with a metric tensor $g(u)$ which is determined arbitrarily, depending on the requirements of the problem we are working out. Certainly, the considered metric tensor has to satisfy the definition properties of the metric tensors determined in paragraph 7. From this point of view, we can speak of a **Riemannian (or pseudo-Riemannian) surface *S***. The properties of a geometric surface arise by the metric tensor defined in the tangent spaces of the surface; by following this way of thinking we say that we construct an "**internal geometry**" of the geometric surface with no reference to any underlying space where it could be immersed.

We start the study of the geometric surfaces by summarizing its fundamental features, ensued by the corresponding concepts we have already used in the description of the surfaces immersed in an underlying 3-dimensional Euclidean space.

A) Tangent spaces of a geometric surface, the metric tensor the basis vector field related with it - Curves in the parameter-space and their images to the geometric surface:

The points of a geometric surface S (simple surface: see paragraph 7) are uniquely determined in a certain system of u-parameters, by the couples $u=(u^1,u^2)$ which are taking values in an open subset B of $\boldsymbol{R}^2$: there is a 1-1 correspondence of B on to the set of the points P_u of the surface. The geometric features of the geometric surface S are specified by the metric tensor $g(u)$, we have imposed on its tangent spaces. The basis-vectors of each tangent space T_uS arise by an abstract basis-vector field $\{e_1(u),e_2(u)\}$ which is compatible with the metric tensor i.e.:

$g_{\mu\nu}(u)=\langle e_\mu(u),e_\nu(u)\rangle,\ \mu,\nu=1,2$

The vectors of any tangent vector space T_uS of the geometric surface S are linear combinations of the basis vectors (elements) of T_uS:

$\xi=e_\mu(u)\xi^\mu$

The real numbers ξ^μ are the components of a tangent vector of some curve in the parameters' space passing by u; i.e. there is always a curve $a^\mu=a^\mu(t)$ of the parameters' domain B such that: $a^\mu(0)=u^\mu$, $\dot{a}^\mu(0)=\xi^\mu$

Remark: We frequently use the following symbolism (see footnote in paragraph 3): Let a be any curve in B passing by the point $u=(u^1,u^2)$, such that $a(0)=u$.

Let $\Delta u=\left(\Delta u^1,\Delta u^2\right)=\left(\dot{a}^1(0),\dot{a}^2(0)\right)\Delta t$, $\Delta t\to 0$ be an infinitesimal tangent vector of a. We symbolize the corresponding infinitesimal vector of the tangent space T_uS:

$\Delta U=e_\mu(u)\Delta u^\mu=e_\mu(u)\dot{a}^\mu(0)\Delta t$

Any curve A on the geometric surface S is the image of a corresponding curve a of the parameters' domain B. Any point $A(t)$ of the curve A is determined by a certain point $a(t)=(a^1(t),a^2(t))$ of a. The tangent vector $\xi_a(t)$ of A at any one of its points $A(t)$ corresponds to the tangent vector $\left(\dot{a}^1(t),\dot{a}^2(t)\right)$ of a, at $a(t)$; it is determined by the analytic expression:

$$\xi_a(t)=e_\mu\left(a(t)\right)\dot{a}^\mu(t)\in T_{a(t)}S \tag{10.2}$$

The set of the tangent vectors $\xi_a(t)$ as t runs its domain, defines a vector field on the geometric surface S.

The elementary length Δs of A at $A(t)$ is calculated by the norm of the infinitesimal tangent vector:

$\Delta A(t)=e_\mu\left(a(t)\right)\dot{a}^\mu(t)\Delta t\in T_{a(t)}S,\ \Delta t\to 0$

$$\Delta s=\sqrt{\langle\Delta A(t),\Delta A(t)\rangle}=\Delta t\sqrt{g_{\mu\nu}\left(a(t)\right)\dot{a}^\mu(t)\dot{a}^\nu(t)} \tag{10.3}$$

B) Consequences of a parameter-transformation to the basis-vector field, the components of the tangent vectors and the metric tensor of a geometric surface:

Consider the diffeomorphic transformation $u^\mu=u^\mu(\tilde{u})$ in the parameters' space. Any point P of S represented by the couple $u=(u^1,u^2)$ in the u-parameters' system, is represented by the ordered pair $\tilde{u}=\left(\tilde{u}^1,\tilde{u}^2\right)$ in the $\tilde{u}$-parameters.

The basis-elements $e_\nu(u),\ \nu=1,2$ of T_uS are being changed to the basis-elements $\tilde{e}_\nu(\tilde{u}),\ \nu=1,2$ in the $\tilde{u}$-parameters.

Any vector ΔU of the tangent space T_uS is expressed as a linear combination of the new basis, according to the relationship:

$$\Delta U = e_\mu(u)\Delta u^\mu = \tilde{e}_\nu(\tilde{u})\Delta\tilde{u}^\nu \underset{def}{=} \Delta\tilde{U} \tag{10.1}$$

The vectors $\Delta U, \Delta\tilde{U}$ are identical (see paragraph 9); the different symbols suggest that they are expressed in different parameters.

The coordinates $\Delta u^1, \Delta u^2$ of the tangent vectors are transformed according to the rule:

$$\Delta u^\mu = \frac{\partial u^\mu}{\partial \tilde{u}^\nu}\Delta\tilde{u}^\nu$$

The coordinates ξ^μ of any vector $\xi = e_\mu(u)\xi^\mu \in T_P S$ are transformed according to the same rule:

$$\xi^\mu = \frac{\partial u^\mu}{\partial \tilde{u}^\nu}\tilde{\xi}^\nu$$

The basis-elements are related through the relationship:

$$e_\mu(u) = \tilde{e}_\nu(\tilde{u})\frac{\partial \tilde{u}^\nu}{\partial u^\mu}$$

The length of any curve *A* in *S* is a geometric invariant; it is not depended on the choice of the parameters' system. We deduce that the elementary length Δs of *A* at anyone of its points is invariant under any parameter-transformation. Hence, the norm of any vector $\xi = e_\mu(u)\xi^\mu \in T_P S,\ P \in S$ is also invariant; as a consequence, the matrix-elements of the metric tensor are being changed as follows:

$$|\xi|^2 = |\tilde{\xi}|^2$$

$$g_{\mu\nu}(u)\xi^\mu\xi^\nu = \tilde{g}_{\kappa\lambda}(\tilde{u})\tilde{\xi}^\kappa\tilde{\xi}^\lambda$$

$$\left(g_{\mu\nu}(u)\frac{\partial u^\mu}{\partial \tilde{u}^\kappa}\frac{\partial u^\nu}{\partial \tilde{u}^\lambda} - \tilde{g}_{\kappa\lambda}(\tilde{u})\right)\tilde{\xi}^\kappa\tilde{\xi}^\lambda = 0$$

$$\tilde{g}_{\kappa\lambda}(\tilde{u}) = g_{\mu\nu}(u)\frac{\partial u^\mu}{\partial \tilde{u}^\kappa}\frac{\partial u^\nu}{\partial \tilde{u}^\lambda} \tag{10.4a}$$

The determinants of the matrices $g = \left[g_{\mu\nu}(u)\right],\ \tilde{g} = \left[\tilde{g}_{\kappa\lambda}(\tilde{u})\right]$ are related with the determinant of the Jacobian matrix of the parameter-transformation, according to the relationship:

$$\det\tilde{g} = \det g\left(\det\left[\frac{\partial u^\mu}{\partial \tilde{u}^\kappa}\right]\right)^2 \tag{10.4b}$$

C) 1-forms on a geometric surface:

The 1-forms on the geometric surface *S* are determined by the usual way (see paragraph 8):

$$\omega_\xi(\zeta) = \langle\xi(u), \zeta\rangle \quad \xi(u), \zeta \in T_u S$$

We symbolize $\xi(u)$ any vector field defined on *S*.

The basic 1-forms are defined on any $T_u S$ according to the relations:

$$\omega_\mu(\zeta) = \langle e_\mu(u), \zeta\rangle = g_{\mu\nu}(u)\zeta^\nu,\ \zeta \in T_u S$$

$$\omega^\mu(\zeta) = \zeta^\mu$$

Hence:

$$\omega_\mu = g_{\mu\nu}(u)\omega^\nu$$

Every 1-form can be expanded as a linear combination of the basic 1-forms:

$$\omega_\xi = \xi^\mu\omega_\mu = g_{\mu\nu}(u)\xi^\mu\omega^\nu$$

Under the parameter-transformation $u^\mu = u^\mu(\tilde{u})$ the basic 1-forms transform according to the following relations (see paragraph 9):

$$\omega^\mu = \frac{\partial u^\mu}{\partial \tilde{u}^\nu}\tilde{\omega}^\nu \tag{10.5a}$$

$$\omega_\mu = \tilde{\omega}_\nu \frac{\partial \tilde{u}^\nu}{\partial u^\mu} \tag{10.5b}$$

Any 1-form $\omega_p = p_\mu(u)\omega^\mu$ is transformed as follows:

$$\omega_p = p_\mu(u)\omega^\mu = p_\mu(u)\frac{\partial u^\mu}{\partial \tilde{u}^\nu}\tilde{\omega}^\nu \underset{def}{=} \tilde{p}_\nu(\tilde{u})\tilde{\omega}^\nu = \tilde{\omega}_{\tilde{p}}$$

$$\tilde{p}_\nu(\tilde{u}) = p_\mu(u)\frac{\partial u^\mu}{\partial \tilde{u}^\nu}$$

D) 2-forms on a geometric surface:

The 2-forms on *S* are the antisymmetric bilinear functions defined on the tangent spaces T_uS of *S* and their values being in $\boldsymbol{R}$ (see paragraph 8):

$$\sigma_u(\xi,\zeta) = \sigma_{\mu\nu}(u)\left(\omega^\mu \wedge \omega^\nu\right)(\xi,\zeta) = \sigma_{\mu\nu}(u)\left(\xi^\mu\zeta^\nu - \xi^\nu\zeta^\mu\right)$$

The real functions $\sigma_{\mu\nu}(u)$ are defined on the parameters' space *B*.

For any geometric surface, the previous relation is simplified as follows:

$$\sigma_u(\xi,\zeta) = \lambda(u)\left(\omega^1 \wedge \omega^2\right)(\xi,\zeta) = \lambda(u)\left(\xi^1\zeta^2 - \xi^2\zeta^1\right),\ \lambda(u) \underset{def}{=} \sigma_{12}(u) - \sigma_{21}(u)$$

$$\sigma_u = \lambda(u)\left(\omega^1 \wedge \omega^2\right)$$

Under the parameter-transformation $u^\mu = u^\mu(\tilde{u})$ the analytic expression of the 2-form σ_u transforms as follows:

$$\sigma_u = \lambda(u)\left(\omega^1 \wedge \omega^2\right) = \lambda(u)\left(\frac{\partial u^1}{\partial \tilde{u}^\kappa}\frac{\partial u^2}{\partial \tilde{u}^\lambda}\tilde{\omega}^\kappa \wedge \tilde{\omega}^\lambda\right) = \lambda(u)\left(\frac{\partial u^1}{\partial \tilde{u}^1}\frac{\partial u^2}{\partial \tilde{u}^2} - \frac{\partial u^1}{\partial \tilde{u}^2}\frac{\partial u^2}{\partial \tilde{u}^1}\right)\tilde{\omega}^1 \wedge \tilde{\omega}^2 =$$

$$= \lambda(u)\det\left[\frac{\partial u^\mu}{\partial \tilde{u}^\kappa}\right]\tilde{\omega}^1 \wedge \tilde{\omega}^2 \underset{def}{=} \tilde{\lambda}(\tilde{u})\tilde{\omega}^1 \wedge \tilde{\omega}^2 = \tilde{\sigma}_{\tilde{u}}$$

$$\tilde{\lambda}(\tilde{u}) = \lambda(u)\det\left[\frac{\partial u^\mu}{\partial \tilde{u}^\kappa}\right] \tag{10.6a}$$

From 10.6a and 10.4b we imply that:

$$\tilde{\lambda}(\tilde{u}) = \lambda(u)\sqrt{\frac{\det \tilde{g}(\tilde{u})}{\det g(u)}}$$

$$\frac{\tilde{\lambda}(\tilde{u})}{\sqrt{\det \tilde{g}(\tilde{u})}} = \frac{\lambda(u)}{\sqrt{\det g(u)}} \tag{10.6b}$$

The 2-form σ_u is invariant under the parameter-transformation $u^\mu = u^\mu(\tilde{u})$ if only the following condition is true:

$$\tilde{\lambda}(\tilde{u}) = \lambda(\tilde{u})$$

In that case according to 10.6b, the quotient $\frac{\lambda(u)}{\sqrt{\det g(u)}}$ is a constant number, independent of the choice of the parameters. As usually (see paragraph 6), we choose the value of the constant equal to 1, and we result that:

$$\lambda(u) = \sqrt{\det g(u)}$$

$$\sqrt{\det g(u)}\left(\omega^1 \wedge \omega^2\right) = \sqrt{\det \tilde{g}(\tilde{u})}\left(\tilde{\omega}^1 \wedge \tilde{\omega}^2\right) \tag{10.7}$$

The 2-form $\sigma_u = \sqrt{g(u)}\left(\omega^1 \wedge \omega^2\right)$ is invariant under any parameter transformation and defines the **area-form** on *S*.

The area of an elementary parallelogram with vertex at the point $P_u \rightleftarrows u = \left(u^1, u^2\right)$ and sides the infinitesimal vectors $\Delta_{(1)}U, \Delta_{(2)}U \in T_u S$ (relations 10.1) is calculated by the expression:

$$\sigma_u\left(\Delta_{(1)}U, \Delta_{(2)}U\right) = \sqrt{\det g(u)}\left(\omega^1 \wedge \omega^2\right)\left(\Delta_{(1)}U, \Delta_{(2)}U\right) \qquad (10.8)$$

11. **Connections on a geometric surface**

Consider any vector field on a geometric surface *S* determined on a certain curve *A* of *S*. *How could one perform a quantitative calculation of the variation of the field along A?* For the case of the Euclidean plane ***E***, the answer to this question is known from the elementary geometry: to determine the variation of a vector field ***F*** along a plane curve *A* from its point *P* to another *Q*, you only have to do two things:

a) Transport parallel to itself the vector ***F***(*P*) along *A*, to the position *Q*. For the Euclidean plane ***E***, the parallel transported vector along any curve *A* is identical to itself; i.e. if we describe the process of the parallel transport by means of a vector-map among the tangent spaces of ***E***, this map is the identity map; by using a symbolic language: name TP_A the parallel-transport-map along the curve *A*:

$PT_A : T_P\boldsymbol{E} \xrightarrow{A} T_Q\boldsymbol{E}$

For any curve *A* joining *P* and *Q*, it holds:

$PT_A\left(\boldsymbol{F}(P)\right) = \boldsymbol{F}(P)$

b) The second thing we have to do is to calculate the difference:

$\boldsymbol{F}(Q) - PT_A\left(\boldsymbol{F}(P)\right) = \boldsymbol{F}(Q) - \boldsymbol{F}(P)$

The previous steps are legitimate because of the identical geometric structure of all the tangent spaces of a Euclidean space. But this is not the case for the geometric surfaces and generally for a Riemannian space. The concept of the "parallel transport" has to be re-defined; this is achieved by introducing the concept of a "connection" on the geometric surface *S*. By defining a connection on *S* we are able to establish an isomorphism between any two tangent spaces of the surface joined by some curve *A*. The isomorphisms induced by the connection along any curve *A* of *S* generate vector fields that determine the "parallel transport on *S*".

In the present paragraph, we define the connection on a geometric surface *S* and the parallel transport of a vector along any curve on *S*. We derive the infinitesimal variation of a vector field along a curve and we introduce the consequent concept of the covariant differentiation. Next, we examine how the connection transforms under a parameter-transformation. Finally, we focus on connections that are "compatible" with the metric tensor defined on the tangent spaces of the geometric surface and we proceed to the calculation of their main features.

Consider a geometric surface *S*. We presume that the set $B \subseteq \boldsymbol{R}^2$ of the parameters $u=(u_1,u_2)$ which determine the points P_u of *S* is a simply connected set: for any two points *u* and *u′* in *B*, we can find a curve $a : u^\mu = a^\mu(t),\ t \in I \subseteq \boldsymbol{R}$ of *B*, connecting the points *u* and *u′*.

For every point $a(t)$ of the curve *a* there is a unique tangent space $T_{a(t)}S$ of the surface *S*. We define a linear map φ^a between the tangent spaces $T_{a(t)}S$ along the curve *a*:

$T_{a(t_0)}S \ni \xi(t_0) \xrightarrow{\varphi^a} \xi(t) \in T_{a(t)}S$

We say that the linear maps φ^a define a **connection** φ on S if they have the following properties:

a) For any u, u' in B and a curve a of B passing by u and u' (say: $u=a(t)$, $u'=a(t')$), connection φ establishes an isomorphism [4], [5], [6] between the tangent spaces $T_{a(t)}S$, $T_{a(t')}S$ of S. We symbolize:

$\varphi^a_{t',t} : T_{a(t)}S \to T_{a(t')}S$

b) For any curve a, the linear map $\varphi^a_{t,t}$ is the identity map:

$\varphi^a_{t,t}\left(\xi(t)\right) = \xi(t),\ \xi(t) \in T_{a(t)}S$

c) For any three points $u=a(t)$, $u'=a(t')$, $u''=a(t'')$ of the curve a, connection φ satisfies the identity:

$\varphi^a_{t',t} = \varphi^a_{t',t''} \circ \varphi^a_{t'',t}$

$\varphi^a_{t',t}\left(\xi(t)\right) = \varphi^a_{t',t''}\left(\varphi^a_{t'',t}\left(\xi(t)\right)\right),\ \xi(t) \in T_{a(t)}S$

Remark: *The matrix-elements of a connection.*

Let $\varphi^a_{t',t} : T_{a(t)}S \to T_{a(t')}S$ be the isomorphism determined by the connection φ between the tangent spaces $T_{a(t)}S$, $T_{a(t')}S$. The matrix $\left[\varphi^{a\mu}_{\nu}\left(t',t\right)\right]$ of the linear map $\varphi^a_{t',t}$ is defined by the relationship:

$$\varphi^a_{t',t}\left(e_\nu\left(u\right)\right) = e_\mu\left(u'\right)\varphi^{a\mu}_{\nu}\left(t',t\right) \tag{11.1}$$

By using the properties a, b and c satisfied by any connection, we are able to verify the following properties of the matrix $\left[\varphi^{a\mu}_{\nu}\left(t',t\right)\right]$ of the connection:

A) $\varphi^{a\mu}_{\nu}\left(t,t\right) = \delta^{\mu}_{\nu}$

B) $\varphi^{a\mu}_{\nu}\left(t',t\right) = \varphi^{a\mu}_{\kappa}\left(t',t''\right)\varphi^{a\kappa}_{\nu}\left(t'',t\right)$

Parallel transport of a vector on a surface along a certain curve
Infinitesimal parallel displacement of a vector on a surface

Consider the curve $a : u_t{}^{\mu} = a^{\mu}(t)$ in the parameters' space B of the geometric surface S. Let φ be a connection on S defined according to the previous section of the present paragraph.

Let $u_0 = \left(a^1(0), a^2(0)\right)$ be an arbitrary point of a, for an appropriate choice of the parameter t. The connection φ determines the isomorphism $\varphi^a_{t,0}$ between the tangent spaces $T_{a(0)}S$, $T_{a(t)}S$ for any value of t. Hence any vector $\xi(0) \in T_{u_0}S$ is mapped by $\varphi^a_{t,0}$ to the vector:

$\xi_{(\varphi)}(t) \in T_{a(t)}S$

$$\xi_{(\varphi)}(t) = \varphi^a_{t,0}\left(\xi(0)\right) \tag{11.2a}$$

$$\xi_{(\varphi)}(0) \underset{def}{=} \xi(0) \tag{11.2b}$$

We say that for any value of t **the vector** $\xi_{(\varphi)}(t) = \varphi^a_{t,0}\left(\xi(0)\right) \in T_{a(t)}S$ **is a parallel transport of the vector** $\xi(0) \in T_{u_0}S$ **with respect to the connection** φ **along the curve** ***a***.

According to 11.2a, we notice that $\varphi^a_{t,0}$ determines a vector field on S along the curve a of the parameters' space. *Is it possible to determine the vector field* $\xi_{(\varphi)}(t)$ *along the curve a, as the solution of a differential equation of the first order with initial condition given by 11.2b?*

Let us study the behavior of the linear map $\varphi^{a}_{t',t}$ between the tangent spaces T_uS, $T_{u'}S$ when u, u' are infinitesimal close to each-other:

$u' = a(t')$, $u = a(t)$, $t' = t + \Delta t$, $\Delta t \to 0$

By keeping terms up to the first order with respect to Δt we obtain the subsequent equations:

$$u' = a(t + \Delta t) = a(t) + \dot{a}(t)\Delta t = u + \Delta u$$

$$\Delta u \underset{def}{=} \dot{a}(t)\Delta t = \left(\dot{a}^1(t), \dot{a}^2(t)\right)\Delta t$$

$$\xi_{(\varphi)}(t + \Delta t) = \varphi^{a}_{t+\Delta t,t}\left(\xi_{(\varphi)}(t)\right), \ \xi_{(\varphi)}(t) = e_\mu(u)\xi_{(\varphi)}{}^{\mu}(t) \in T_uS, \ \xi_{(\varphi)}(t + \Delta t) \in T_{u+\Delta u}S$$

$$e_\mu(u + \Delta u)\xi_{(\varphi)}{}^{\mu}(t + \Delta t) = \varphi^{a}_{t+\Delta t,t}\left(e_\nu(u)\right)\xi_{(\varphi)}{}^{\nu}(t)$$

Then, by using 11.1 we are getting:

$$e_\mu(u + \Delta u)\xi_{(\varphi)}{}^{\mu}(t + \Delta t) = e_\mu(u + \Delta u)\varphi^{a\mu}_{\nu}(t + \Delta t, t)\xi_{(\varphi)}{}^{\nu}(t)$$

$$\xi_{(\varphi)}{}^{\mu}(t + \Delta t) = \varphi^{a\mu}_{\nu}(t + \Delta t, t)\xi_{(\varphi)}{}^{\nu}(t) \tag{11.3}$$

We expand the quantities with argument $t + \Delta t$ in Taylor series and keep terms up to the first order. Then, by using the properties of the matrices $\left[\varphi^{a\mu}_{\nu}(t', t)\right]$ referred in the previous section, we obtain the equation:

$$\frac{d\xi_{(\varphi)}{}^{\mu}(t)}{dt} = \Gamma^{\mu}_{\nu\kappa}\left(a(t)\right)\dot{a}^{\kappa}(t)\xi_{(\varphi)}{}^{\nu}(t) \tag{11.4}$$

We symbolize:

$$\bar{\varphi}^{a\mu}_{\nu}(u', u) = \bar{\varphi}^{a\mu}_{\nu}\left(a(t'), a(t)\right) \underset{def}{=} \varphi^{a\mu}_{\nu}(t', t)$$

$$\Gamma^{\mu}_{\nu\kappa}(u) \underset{def}{=} \left.\frac{\partial\bar{\varphi}^{a\mu}_{\nu}(u', u)}{\partial u'^{\kappa}}\right|_{u'=u}$$

The symbols $\Gamma^{\mu}_{\nu\kappa}(u)$ are known as "**Christoffel symbols**" [1], [3].

The vector field $\xi_{(\varphi)}$ is the solution of 11.4, with initial condition determined in 11.2b. Obviously, one could solve equation 11.4 only if the analytic expressions of the Christoffel symbols $\Gamma^{\mu}_{\nu\kappa}(u)$ and of the curve a are known.

Remarks:

A) The differential equation 11.4 can also be written as follows:

$$\frac{d\xi_{(\varphi)}{}^{\mu}(t)}{dt} = \gamma^{a\mu}_{\nu}(t)\xi_{(\varphi)}{}^{\nu}(t) \tag{11.5}$$

$$\gamma^{a\mu}_{\nu}(t) \underset{def}{=} \left.\frac{d\varphi^{a\mu}_{\nu}(t', t)}{dt'}\right|_{t'=t}$$

B) The solution of 11.4 or 5 is a vector field on S defined along the curve a of the parameters' space; the members of the vector field are parallel to the initial vector $\xi(0) \in T_{u_0}S$ and are determined by the relationships 11.2a and b:

$$\xi_{(\varphi)}(t) = \varphi^{a}_{t,0}\left(\xi(0)\right)$$

According to the properties of the connection, for any two vectors of the field, it holds that:

$$\xi_{(\varphi)}(t') = \varphi^{a}_{t',t}\left(\xi_{(\varphi)}(t)\right)$$

We infer that all the vectors of the field are parallel to each-other; by using the properties of the connection we verify that the relation of "parallelism" is an **equivalence relation**: a)

any member is parallel to itself; b) if the member U is parallel to W, then W is parallel to U; c) if U is parallel to W and W to V, then U is parallel to V:

Steps to the proof

a) $\xi_{(\varphi)}(t) = \varphi^{a}_{t,t}\left(\xi_{(\varphi)}(t)\right)$

b) $\xi_{(\varphi)}(t') = \varphi^{a}_{t',t}\left(\xi_{(\varphi)}(t)\right) \Rightarrow \xi_{(\varphi)}(t) = \varphi^{a}_{t,t'}\left(\xi_{(\varphi)}(t')\right)$

c) $\xi_{(\varphi)}(t) = \varphi^{a}_{t,t'}\left(\xi_{(\varphi)}(t')\right)$ AND $\xi_{(\varphi)}(t') = \varphi^{a}_{t',t''}\left(\xi_{(\varphi)}(t'')\right)$

Then:

$$\xi_{(\varphi)}(t) = \varphi^{a}_{t,t'}\left(\xi_{(\varphi)}(t')\right) = \varphi^{a}_{t,t'}\left(\varphi^{a}_{t',t''}\left(\xi_{(\varphi)}(t'')\right)\right) = \varphi^{a}_{t,t''}\left(\xi_{(\varphi)}(t'')\right)$$

The vector field that is parallel to some vector along a curve a, is sometimes called "**a vector field parallel to itself**" or "**parallel vector field along the curve *a***".

Example 11A
Examples of connections in the Euclidean plane - Application of the equation 11.5

In the first section of the present example we are getting some information and formalism related with the plane curves. Then, in the second and third sections we define two different connections in the Euclidean plane. They are briefly described as follows:
A) According to the first connection the isomorphism between any two tangent spaces at the points A and B connected by a curve a, is determined so that any tangent vector U_A at A is mapped to U_B at B, where U_B is the rotation of U_A at an angle equal to the angle formed by the tangents of the curve a at A and B (figure 11.1). One well-known result issued by this connection is the Frenet - Serret formulas for the plane curves.
B) The second connection is the used in the elementary geometry of the Euclidean plane: The parallel displacement along any curve is defined so that the Cartesian coordinates of every tangent vector and its image are identical.

1) About curves in the Euclidean plane

Consider the Euclidean plane $\boldsymbol{R}^2_0$ and any curve $a: u^{\mu} = a^{\mu}(s)$ in it. Let u^1, u^2 be the Cartesian coordinates of the points $u = (u^1, u^2) \in \boldsymbol{R}^2_0$ and $\{e_1, e_2\}$ the corresponding "natural" basis of the plane. The parameter s is the length of the curve from an arbitrary point $a(0)$ on it:

$$s = \int_0^s \left(\dot{a}(\sigma)\cdot\dot{a}(\sigma)\right)^{1/2} d\sigma \qquad \text{(E11A.1)}$$

We symbolize:

$$\dot{a}(\sigma) \underset{def}{=} \left(\dot{a}^1(\sigma), \dot{a}^2(\sigma)\right) \underset{def}{=} \left(\frac{da^1(\sigma)}{d\sigma}, \frac{da^2(\sigma)}{d\sigma}\right)$$

The plane $\boldsymbol{R}^2_0$ is equipped with the Euclidean inner product; hence:

$$\dot{a}(\sigma)\cdot\dot{a}(\sigma) = \delta_{\mu\nu}\dot{a}^{\mu}(\sigma)\dot{a}^{\nu}(\sigma)$$

From E11A.1 we imply that

$$\dot{a}(s)\cdot\dot{a}(s) = 1 \qquad \text{(E11A.2)}$$

The vector $\dot{a}(s)$ is tangent to the curve at its point $u = a(s)$ and belongs to the tangent space $T_u\boldsymbol{R}^2_0$ of the plane.

Given that our plane is Euclidean, every tangent space $T_u\boldsymbol{R}_0^2$ is also Euclidean and in the system of the Cartesian coordinates u^1, u^2 its basis-vectors are the constant, independent of u vectors e_1, e_2. The inner product at every tangent space $T_u\boldsymbol{R}_0^2$ is determined by the Euclidean metric tensor:

$\delta_{\mu\nu} = e_\mu \cdot e_\nu \quad \mu,\nu = 1,2$

Notice that for any two different points $u=a(s)$ and $u'=a(s')$ of the curve, the corresponding tangent vectors are written as linear combinations of the natural basis-vectors:

$\dot{a}(s) = e_\mu \dot{a}^\mu(s)$

$\dot{a}(s') = e_\mu \dot{a}^\mu(s')$

We see that in that case it is legitimate to transport each vector belonging to any tangent space of the Euclidean plane parallel to itself without changing it, to any other point of the plane (figure 11.1).

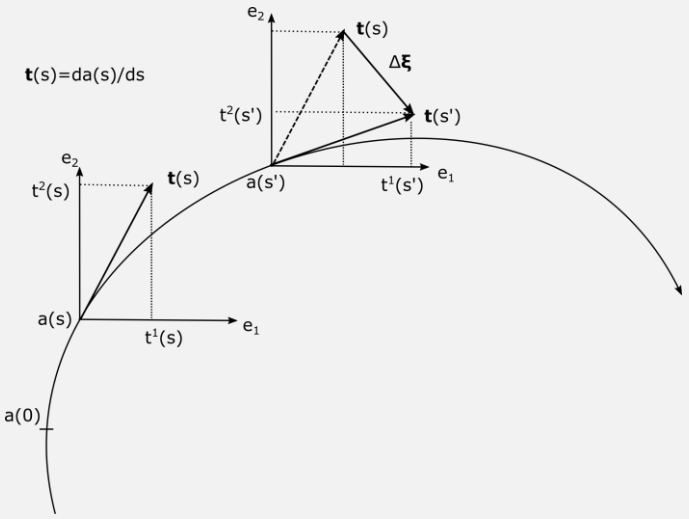

Figure 11.1: Parallel displacement in the Euclidean plane.

At every point $u=a(s)$ of the curve we are possible to define a unit vector $n_a(s) \in T_u\boldsymbol{R}_0^2$ which is normal to the tangent vector $\dot{a}(s)$ i.e.:

$$n_a(s) = \left(\dot{a}^2(s), -\dot{a}^1(s)\right) = e_1\dot{a}^2(s) - e_2\dot{a}^1(s) \qquad \text{(E11A.3)}$$

From E11A.3 we are getting the identities:

$n_a(s)\cdot n_a(s) = 1$, $n_a(s)\cdot \dot{a}(s) = 0$

We define the radius of curvature $r(s)$ and the curvature $\kappa(s)$ of the curve a at s by the relationships:

$$\kappa(s) \underset{def}{=} \frac{1}{r(s)} \underset{def}{=} \lim_{\Delta s\to 0}\left(\frac{\Delta\theta}{\Delta s}\right)$$

The infinitesimal quantity Δs is the length of the part of the curve between its neighboring points:

$u = a(s)$, $u' = a(s+\Delta s) = u + \dot{a}(s)\Delta s$, $\Delta s \to 0$

The angle $\Delta\theta$ is determined by the vectors (figure 11.2):

$\boldsymbol{t}(s) = \dot{a}(s)$, $\boldsymbol{t}(s') = \dot{a}(s')$, $s' = s + \Delta s$

2) Definition of a connection in the Euclidean plane

Consider again the curve $a: u^\mu = a^\mu(s)$ of the Euclidean plane and any vector:

$\xi(0) \in T_{a(0)}\boldsymbol{R}_0^2$

We are going to define a connection φ in $\boldsymbol{R}_0^2$ by establishing an isomorphism among the tangent spaces of the Euclidean plane, along the curve:

$\varphi_{s',s}^a : T_{a(s)}\boldsymbol{R}_0^2 \to T_{a(s')}\boldsymbol{R}_0^2$

Let us consider two neighboring points of the curve *a*:

$u = a(s),\ u + \Delta u = a(s + \Delta s),\ \Delta s \to 0$

The vector $\xi_{(\varphi)}(s) = \varphi_{s,0}^a\left(\xi(0)\right) \in T_u\boldsymbol{R}_0^2$ is mapped by $\varphi_{s+\Delta s,s}^a$ to the vector:

$\xi_{(\varphi)}(s + \Delta s) = \varphi_{s+\Delta s,s}^a\left(\xi(s)\right) \in T_{u+\Delta u}\boldsymbol{R}_0^2$

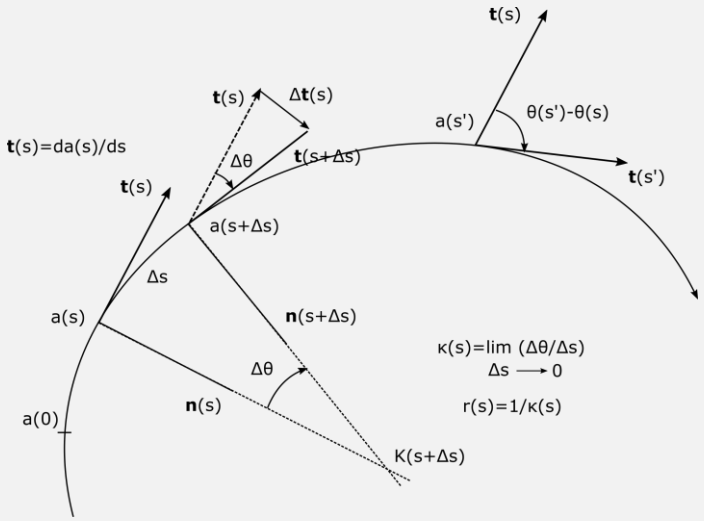

Figure 11.2: Tangent and normal vectors at the points of a plane curve. Curvature and radius of curvature at the points of a plane curve. The angle θ is determined as the angle formed between the tangent vectors **t**(s) and **t**(0). The parameter s is the length of the curve measured by a specified point a(0) of the curve.

We presume that for any curve *a*, $\varphi_{s+\Delta s,s}^a$ is an infinitesimal rotation in the Euclidean plane, with angle of revolution:

$\Delta\theta = \kappa(s)\,\Delta s$

Where: $\kappa(s)$ is the curvature of *a* at its point $u = a(s)$

This implies that $\varphi_{s+\Delta s,s}^a$ can be written in the form:

$$\varphi_{s+\Delta s,s}^a = \mathrm{Id} + \kappa(s)\,\Delta s \cdot \Omega \qquad \text{(E11A.4)}$$

We symbolize Id the identity map; the linear map Ω is to be determined by our assumption that $\xi_{(\varphi)}(s + \Delta s)$ and $\xi_{(\varphi)}(s)$ have equal lengths: the following identities are valid:

$\xi_{(\varphi)}(s + \Delta s) \cdot \xi_{(\varphi)}(s + \Delta s) = \xi_{(\varphi)}(s) \cdot \xi_{(\varphi)}(s)$

$\left(\xi_{(\varphi)}(s) + \Delta\theta \cdot \Omega\left(\xi_{(\varphi)}(s)\right)\right) \cdot \left(\xi_{(\varphi)}(s) + \Delta\theta \cdot \Omega\left(\xi_{(\varphi)}(s)\right)\right) = \xi_{(\varphi)}(s) \cdot \xi_{(\varphi)}(s)$

$\xi_{(\varphi)}(s) \cdot \Omega\left(\xi_{(\varphi)}(s)\right) + \Omega\left(\xi_{(\varphi)}(s)\right)\xi_{(\varphi)}(s) = 0$

$$\xi_{(\varphi)}(s) \cdot \left(\Omega\left(\xi_{(\varphi)}(s)\right) + \Omega^T\left(\xi_{(\varphi)}(s)\right)\right) = 0 \qquad \text{(E11A.5)}$$

The linear map Ω^T is the transpose of Ω defined by the relationship:

$\xi \cdot \Omega(\xi') = \Omega^T(\xi) \cdot \xi',\ \ \xi, \xi' \in T_u\boldsymbol{R}_0^2$

Equation E11A.5 holds for any vector: $\xi_{(\varphi)}(s) = \varphi_{s,0}^a\left(\xi(0)\right) \in T_u\boldsymbol{R}_0^2$

We infer that:

$$\Omega\left(\xi_{(\varphi)}(s)\right) + \Omega^{T}\left(\xi_{(\varphi)}(s)\right) = 0 \qquad \text{(E11A.6)}$$

The matrix $\left[\Omega_{\mu}^{\nu}\right]$ of Ω is defined by the relationship:

$\Omega\left(e_{\mu}\right) = e_{\nu}\,\Omega_{\mu}^{\nu}$

From E11A.6, we imply that the matrix-elements of $\left[\Omega_{\mu}^{\nu}\right]$ satisfy the equations:

$\Omega_{\mu}^{\nu} + \Omega^{T\nu}_{\ \ \mu} = 0$

A non-trivial solution of the previous equations is expressed by the following choice:

$$\left[\Omega_{\mu}^{\nu}\right] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Hence, the matrix-elements of $\varphi^{a}_{s+\Delta s,s}$ are obtained from E10A.4:

$$\varphi^{a\mu}_{\ \ \nu}\left(s+\Delta s, s\right) = \delta_{\nu}^{\mu} + \kappa(s)\,\Delta s\,\Omega_{\nu}^{\mu} \qquad \text{(E11A.7)}$$

From 11.3 and E11A.7 we derive the consecutive equations:

$\xi_{(\varphi)}(s+\Delta s) = \varphi^{a}_{s+\Delta s,s}\left(\xi(s)\right)$

$\xi^{\mu}_{(\varphi)}(s+\Delta s) = \varphi^{a\mu}_{\ \ \nu}\left(s+\Delta s, s\right)\xi^{\nu}_{(\varphi)}(s)$

$\xi^{\mu}_{(\varphi)}(s+\Delta s) = \xi^{\mu}_{(\varphi)}(s) + \kappa(s)\,\Delta s\,\Omega_{\nu}^{\mu}\,\xi^{\nu}_{(\varphi)}(s)$

$$\frac{d\xi^{\mu}_{(\varphi)}(s)}{ds} = \kappa(s)\,\Omega_{\nu}^{\mu}\,\xi^{\nu}_{(\varphi)}(s) \qquad \text{(E11A.8)}$$

With initial condition: $\xi^{\mu}_{(\varphi)}(0) = \xi^{\mu}(0)$

A comparison of E11A.8 with equation 11.5 implies that in our case the values of the symbols $\gamma^{a\mu}_{\ \ \nu}(s)$ are given by the relationships:

$\gamma^{a\mu}_{\ \ \nu}(s) = \kappa(s)\,\Omega_{\nu}^{\mu}$

Let us try to solve the differential equations E11A.8:

We apply the definition of the curvature $\kappa(s)$ according to the preceding section of the present example:

$$\kappa(s) = \frac{d\theta}{ds} \qquad \text{(E11A.9a)}$$

We choose the parameters *s* and θ so that for the initial point *s*=0 the value of θ is zero (figure 11.2): $\theta(0) = 0$

Then, by E11A.9a we imply that:

$$\theta(s) = \int_{0}^{s} \kappa(\sigma)\,d\sigma \qquad \text{(E11A.9b)}$$

We define: $\bar{\xi}_{(\varphi)}(\theta) \underset{def}{=} \xi_{(\varphi)}\left(s(\theta)\right)$

Equation E11A.8 leads to the following matrix differential equation

$$\frac{d\bar{\xi}_{(\varphi)}(\theta)}{d\theta} = \Omega\,\bar{\xi}_{(\varphi)}(\theta) \qquad \text{(E11A.10)}$$

$$\bar{\xi}_{(\varphi)}(\theta) = \begin{bmatrix} \bar{\xi}^{1}_{(\varphi)}(\theta) \\ \bar{\xi}^{2}_{(\varphi)}(\theta) \end{bmatrix}$$

The solution of E11A.10 with initial condition $\bar{\xi}_{(\varphi)}(0) = \bar{\xi}(0)$ is obtained by expanding $\bar{\xi}_{(\varphi)}(\theta)$ in a Taylor series; we obtain:

$$\bar{\xi}_{(\varphi)}(\theta)=\left(\mathrm{I}+\frac{\theta}{1!}\Omega+\frac{\theta^2}{2!}\Omega^2+\ldots\right)\bar{\xi}(0)=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\bar{\xi}(0) \qquad \text{(E11A.11)}$$

$$\bar{\xi}_{(\varphi)}(\theta')=\begin{pmatrix}\cos(\theta'-\theta) & \sin(\theta'-\theta)\\ -\sin(\theta'-\theta) & \cos(\theta'-\theta)\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\bar{\xi}(0)=\begin{pmatrix}\cos(\theta'-\theta) & \sin(\theta'-\theta)\\ -\sin(\theta'-\theta) & \cos(\theta'-\theta)\end{pmatrix}\bar{\xi}_{(\varphi)}(\theta)$$

Hence the matrix of the connection φ is explicitly given by the relationship:

$$\left[\varphi^{a\mu}{}_{\nu}(s',s)\right]=\begin{pmatrix}\cos\left(\int_s^{s'}\kappa(\sigma)d\sigma\right) & \sin\left(\int_s^{s'}\kappa(\sigma)d\sigma\right)\\ -\sin\left(\int_s^{s'}\kappa(\sigma)d\sigma\right) & \cos\left(\int_s^{s'}\kappa(\sigma)d\sigma\right)\end{pmatrix} \qquad \text{(E11A.12)}$$

If we reflect on figure 11.2, we notice that the tangent vectors $\dot{a}(s)$ and $\dot{a}(s')$ of the curve *a* have both unit length and they form an angle:

$$\Delta\theta=\theta(s')-\theta(s)=\int_s^{s'}\kappa(\sigma)d\sigma$$

Hence $\dot{a}(s)$ and $\dot{a}(s')$ are related via a rotation operator whose matrix is identical to the matrix $\left[\varphi^{a\mu}{}_{\nu}(s',s)\right]$ of the connection.

We choose:

$\xi_{(\varphi)}(0)=\dot{a}(s)\big|_{s=0}\in T_{a(0)}\boldsymbol{R}_0^2$

Then:

$\xi_{(\varphi)}(s)=\varphi^{a}_{s,0}\left(\dot{a}(0)\right)=\dot{a}(s)\in T_{a(s)}\boldsymbol{R}_0^2$

Hence, the coordinates of $\dot{a}(s)$ are solutions of the equations E11A.8:

$$\frac{d\dot{a}^1(s)}{ds}=\kappa(s)\dot{a}^2(s),\ \frac{d\dot{a}^2(s)}{ds}=-\kappa(s)\dot{a}^1(s)$$

By using the definition of the unit (principal) normal *n*(*s*) of the curve at *s*, the above equations take the form:

$$\frac{d\dot{a}(s)}{ds}=\kappa(s)n(s) \qquad \text{(E11A.13a)}$$

$$\frac{dn(s)}{ds}=-\kappa(s)\dot{a}(s) \qquad \text{(E11A.13b)}$$

Equations E11A.13a and b are known as "the Frenet - Serret formulas" for the case of any curve lying on the Euclidean plane.

3) Parallel displacement of a vector in the Euclidean plane

Another connection that we could define on the Euclidean plane is the rather trivial one: the identity map. We have seen that in Cartesian coordinates the basis-elements of the tangent spaces of the Euclidean plane are independent of their position *u*. For the tangent space $T_u\boldsymbol{R}_0^2$ the basis-elements are determined as the tangents of the following curves, at *t*=0:

$b_{(1)}(t)=\left(u^1+t,u^2\right)$, $b_{(2)}(t)=\left(u^1,u^2+t\right)$

$b_{(1)}(0)=b_{(2)}(0)=\left(u^1,u^2\right)$

$$e_1=\frac{d}{dt}b_{(1)}(t)\Big|_{t=0}=(1,0),\ e_2=\frac{d}{dt}b_{(2)}(t)\Big|_{t=0}=(0,1)$$

In Cartesian coordinates, the metric tensor in every tangent plane $T_u\boldsymbol{R}_0^2$ of the Euclidean plane is determined by the relations:

$g_{\mu\nu} = e_\mu \cdot e_\nu = \delta_{\mu\nu}$

The basis vectors and the metric tensor are independent of *u*. So, we presume that the matrix of the connection φ for any curve *a* of the plane is the identity matrix:

$\varphi^{(a)\mu}_{\ \ \nu}(t', t) = \delta^\mu_\nu$

The defined connection φ establishes an isomorphism between any couple of tangent spaces of the Euclidean plane, connected with a curve *a*:

$T_{a(0)}\boldsymbol{R}^2_0 \to T_{a(t)}\boldsymbol{R}^2_0$

Any vector $\xi \in T_{a(0)}\boldsymbol{R}^2_0$ is mapped to the vector: $\xi_{(\varphi)}(t) = \varphi^a{}_{t,0}(\xi) \in T_{a(t)}\boldsymbol{R}^2_0$

The vectors ξ and $\xi_{(\varphi)}(t)$ have equal coordinates with respect to the "natural" basis $\{e_1, e_2\}$ and, consequently the same length. We say that the vector field $\xi_{(\varphi)}(t)$ is the **parallel displacement** of ξ along the curve *a*. This is the concept of the parallel transport used in the elementary plane Geometry.

Covariant differentiation on a geometric surface

Consider a vector field $\bar{\xi}(u)$ defined on the tangent spaces of the geometric surface *S*. *Is it possible to evaluate the infinitesimal variation of* $\bar{\xi}(u)$ *when we move from the point* P_u *to the point* $P_{u+\Delta u}$, $\Delta u \to (0,0)$ *of S?* The difficulty is focused on the fact that the vectors $\bar{\xi}(u)$ and $\bar{\xi}(u+\Delta u)$ belong to different vector spaces:

$\bar{\xi}(u) \in T_u S$, $\bar{\xi}(u+\Delta u) \in T_{u+\Delta u} S$

How shall we compare $\bar{\xi}(u)$ *with* $\bar{\xi}(u+\Delta u)$ *in order to define their variation*?

Thanks to the concept of the connection, we can "transfer" one of the vectors, say $\bar{\xi}(u+\Delta u)$ to the neighboring space $T_u S$ and define the variation $D_{\Delta u}\bar{\xi}(u)$ of the vector field along Δu by the relationship:

$$D_{\Delta u}\bar{\xi}(u) = \bar{\varphi}_{u,u+\Delta u}\left(\bar{\xi}(u+\Delta u)\right) - \bar{\xi}(u) = e_\mu(u) D^\mu_{\Delta u}\bar{\xi}(u) \qquad (11.6)$$

The vector $D_{\Delta u}\bar{\xi}(u) \in T_u S$ is called "**the covariant differential**" of the vector field $\bar{\xi}(u)$ along the infinitesimal tangent vector $\Delta u = \dot{a}(t)\Delta t$, $\Delta t \to 0$ of some curve $a: u^\mu = a^\mu(t)$ defined in the parameters' space *B*.

The "**covariant derivative**" $\dfrac{D\xi(t)}{dt}$ of the vector field $\xi(t) \underset{def}{=} \bar{\xi}(a(t))$ along the curve *a*, is evaluated by the equation:

$$\frac{D\xi(t)}{dt} = \frac{D_{\Delta u}\bar{\xi}(a(t))}{dt} = \lim_{\Delta t \to 0} \frac{\bar{\varphi}_{u,u+\Delta u}\left(\bar{\xi}(u+\dot{a}(t)\Delta t)\right) - \bar{\xi}(u)}{\Delta t} \qquad (11.7)$$

Let us calculate the coordinates $D^\mu_{\Delta u}\bar{\xi}(u)$ of the covariant differential $D_{\Delta u}\bar{\xi}(u)$ related to the basis vectors $e_\mu(u)$, $\mu = 1,2$ of the tangent space. By using relation 11.6 and the properties of the connections, and by keeping terms up to the first order with respect to Δu^μ we obtain:

$e_\mu(u) D^\mu_{\Delta u}\bar{\xi}(u) = e_\mu(u)\left(\bar{\varphi}^\mu_\nu(u, u+\Delta u)\bar{\xi}^\nu(u+\Delta u) - \bar{\xi}^\mu(u)\right)$

$$D^{\mu}_{\Delta u}\bar{\xi}(u)=\left(\bar{\varphi}^{\mu}_{v}(u,u)+\left.\frac{\partial\bar{\varphi}^{\mu}_{v}(u,u')}{\partial u'^{k}}\right|_{u'=u}\Delta u^{\kappa}\right)\left(\bar{\xi}^{v}(u)+\frac{\partial\bar{\xi}^{v}(u)}{\partial u^{\kappa}}\Delta u^{\kappa}\right)-\bar{\xi}^{\mu}(u)$$

$$D^{\mu}_{\Delta u}\bar{\xi}(u)=\left(\frac{\partial\bar{\xi}^{\mu}(u)}{\partial u^{\kappa}}+\left.\frac{\partial\bar{\varphi}^{\mu}_{v}(u,u')}{\partial u'^{k}}\right|_{u'=u}\bar{\xi}^{v}(u)\right)\Delta u^{\kappa}$$

We define the Christoffel symbols:

$$\bar{\Gamma}^{\mu}_{v\kappa}(u)\underset{def}{=}\left.\frac{\partial\bar{\varphi}^{\mu}_{v}(u,u')}{\partial u'^{k}}\right|_{u'=u} \tag{11.8}$$

The components of the covariant differential and the covariant derivative of the vector field $\bar{\xi}(u)$ along the curve *a*, are expressed respectively by the consequent relationships:

$$D^{\mu}_{\Delta u}\bar{\xi}(u)=\left(\frac{\partial\bar{\xi}^{\mu}(u)}{\partial u^{\kappa}}+\bar{\Gamma}^{\mu}_{v\kappa}(u)\bar{\xi}^{v}(u)\right)\Delta u^{\kappa} \tag{11.9a}$$

$$\frac{D^{\mu}_{(a)}\xi(t)}{dt}=\frac{d\xi^{\mu}(t)}{dt}+\bar{\Gamma}^{\mu}_{v\kappa}\left(u(t)\right)\xi^{v}(t)\frac{da^{\kappa}(t)}{dt} \tag{11.9b}$$

Remarks:

A) We notice that the covariant derivative of the vector field $\bar{\xi}(u)$ is determined by the Christoffel symbols $\bar{\Gamma}^{\mu}_{v\kappa}(u)$ which have been defined by 11.8. However, we have also defined (relation 11.4) the quantities:

$$\Gamma^{\mu}_{v\kappa}(u)\underset{def}{=}\left.\frac{\partial\bar{\varphi}^{\mu}_{v}(u',u)}{\partial u'^{k}}\right|_{u'=u}$$

The quantities $\Gamma^{\mu}_{v\kappa}(u)$ are also considered as "Christoffel symbols". There is a close relation between $\Gamma^{\mu}_{v\kappa}(u)$, $\bar{\Gamma}^{\mu}_{v\kappa}(u)$ which is derived as follows:

From the definition-properties of a connection, we imply that:

$$\bar{\varphi}_{u,u+\Delta u}\left(\bar{\varphi}_{u+\Delta u,u}\left(\bar{\xi}(u)\right)\right)=\bar{\xi}(u) \tag{11.10}$$

$$\bar{\xi}(u)=e_{v}(u)\bar{\xi}^{v}(u)\in T_{u}S$$

Consequently, from 11.10 we imply the relationships:

$$\bar{\varphi}_{u,u+\Delta u}\left(\bar{\varphi}_{u+\Delta u,u}\left(e_{\mu}(u)\right)\right)\bar{\xi}^{\mu}(u)=e_{\mu}(u)\bar{\xi}^{\mu}(u)$$

$$\bar{\varphi}_{u,u+\Delta u}\left(\bar{\varphi}_{u+\Delta u,u}\left(e_{\mu}(u)\right)\right)=e_{\mu}(u)$$

$$\bar{\varphi}_{u,u+\Delta u}\left(e_{\kappa}(u+\Delta u)\bar{\varphi}^{\kappa}_{\mu}(u+\Delta u,u)\right)=e_{\mu}(u)$$

$$\bar{\varphi}_{u,u+\Delta u}\left(e_{\kappa}(u+\Delta u)\right)\bar{\varphi}^{\kappa}_{\mu}(u+\Delta u,u)=e_{\mu}(u)$$

$$e_{v}(u)\bar{\varphi}^{v}_{\kappa}(u,u+\Delta u)\bar{\varphi}^{\kappa}_{\mu}(u+\Delta u,u)=\delta^{v}_{\mu}e_{v}(u)$$

$$\bar{\varphi}^{v}_{\kappa}(u,u+\Delta u)\bar{\varphi}^{\kappa}_{\mu}(u+\Delta u,u)=\delta^{v}_{\mu}$$

We expand in Taylor series and keep terms up to the first order with respect to the infinitesimal quantities:

$$\left(\delta^{v}_{\kappa}+\left.\frac{\partial\bar{\varphi}^{v}_{\kappa}(u,u')}{\partial u'^{\lambda}}\right|_{u'=u}\Delta u^{\lambda}\right)\left(\delta^{\kappa}_{\mu}+\left.\frac{\partial\bar{\varphi}^{\kappa}_{\mu}(u',u)}{\partial u'^{\lambda}}\right|_{u'=u}\Delta u^{\lambda}\right)=\delta^{v}_{\mu}$$

$$\left(\left.\frac{\partial\bar{\varphi}^{v}_{\mu}(u',u)}{\partial u'^{\lambda}}\right|_{u'=u}+\left.\frac{\partial\bar{\varphi}^{v}_{\mu}(u,u')}{\partial u'^{\lambda}}\right|_{u'=u}\right)\Delta u^{\lambda}=0$$

Hence:

$$\left.\frac{\partial \bar{\varphi}^{\nu}_{\mu}(u',u)}{\partial u'^{\lambda}}\right|_{u'=u} + \left.\frac{\partial \bar{\varphi}^{\nu}_{\mu}(u,u')}{\partial u'^{\lambda}}\right|_{u'=u} = 0$$

$$\Gamma^{\nu}_{\mu\lambda}(u) + \bar{\Gamma}^{\nu}_{\mu\lambda}(u) = 0 \qquad (11.11a)$$

B) We define the quantities:

$\Gamma_{\kappa\mu\lambda}(u) \underset{def}{=} g_{\kappa\nu}(u)\Gamma^{\nu}_{\mu\lambda}(u)$, $\bar{\Gamma}_{\kappa\mu\lambda}(u) \underset{def}{=} g_{\kappa\nu}(u)\bar{\Gamma}^{\nu}_{\mu\lambda}(u)$

The matrix $g(u) = \left[g_{\kappa\mu}(u)\right]$ is the matrix of the metric tensor expressed in the u-parameters, on the tangent space T_uS and $g^{-1}(u) = \left[g^{\nu\kappa}(u)\right]$ its inverse:

$g^{\nu\kappa}(u)g_{\kappa\mu}(u) = \delta^{\nu}_{\mu}$

Hence:

$\Gamma^{\nu}_{\mu\lambda}(u) = g^{\nu\kappa}(u)\Gamma_{\kappa\mu\lambda}(u)$

$\bar{\Gamma}^{\nu}_{\mu\lambda}(u) = g^{\nu\kappa}(u)\bar{\Gamma}_{\kappa\mu\lambda}(u)$

From (10.11a) we imply that:

$$\Gamma_{\kappa\mu\lambda}(u) + \bar{\Gamma}_{\kappa\mu\lambda}(u) = 0 \qquad (11.11b)$$

All these quantities are called "Christoffel symbols".

How do the Christoffel symbols transform under a parameter transformation?

Consider an infinitesimal vector ΔU in the tangent space T_uS of the geometric surface S:

$\Delta U = e_{\mu}(u)\Delta u^{\mu} \in T_uS$

The infinitesimal vector $\Delta u = \left(\Delta u^1, \Delta u^2\right) = \left(\dot{a}^1(t), \dot{a}^1(t)\right)\Delta t$, $\Delta t \to 0$ is tangent of the curve $a : u = a(t)$ which is defined in the domain $B \subseteq \boldsymbol{R}^2$ of the u-parameters.

Let us apply a transformation $u^{\mu} = u^{\mu}(\tilde{u})$ in the parameters' space B. As usually, we assume that a parameter transformation is invertible, differentiable at least up to the second order, and its invert transformation is also differentiable at least up to the second order. As we have already noticed (paragraphs 5, 9, 10) the points P of the geometric surface do not change under any parameters' transformation:

$P \equiv P_u \equiv P_{\tilde{u}}$

The same is true for the vectors of the tangent spaces T_PS of S:

$\Delta U = e_{\mu}(u)\Delta u^{\mu} = \tilde{e}_{\nu}(\tilde{u})\Delta\tilde{u}^{\nu} = \Delta\tilde{U}$

We have seen (see paragraph 10) that under the transformation $u^{\mu} = u^{\mu}(\tilde{u})$ the coordinates Δu^{μ} of the tangent vector ΔU and the basis vectors transform according to the relations:

$\Delta u^{\mu} = J^{\mu}_{\nu}(\tilde{u})\Delta\tilde{u}^{\nu}$, $\Delta\tilde{u}^{\mu} = \tilde{J}^{\mu}_{\nu}(u)\Delta u^{\nu}$, $J^{\mu}_{\nu}(\tilde{u}) \underset{def}{=} \frac{\partial u^{\mu}}{\partial\tilde{u}^{\nu}}$, $\tilde{J}^{\mu}_{\nu}(u) \underset{def}{=} \frac{\partial\tilde{u}^{\mu}}{\partial u^{\nu}}$

$\tilde{e}_{\nu}(\tilde{u}) = e_{\mu}(u)J^{\mu}_{\nu}$, $e_{\nu}(u) = \tilde{e}_{\mu}(\tilde{u})\tilde{J}^{\mu}_{\nu}$

For any vector field $\xi(u) = e_{\mu}(u)\xi^{\mu}(u) \in T_uS$ we have:

$e_{\mu}(u)\xi^{\mu}(u) = \tilde{e}_{\nu}(\tilde{u})\tilde{\xi}^{\nu}(\tilde{u})$

$\xi^{\mu}(u) = J^{\mu}_{\nu}\tilde{\xi}^{\nu}(\tilde{u})$

Let us consider the connection $\bar{\varphi}$ defined on the geometric surface *S*. We shall examine how the matrix $\left[\bar{\varphi}_{\mu}^{\nu}(u',u)\right]$, $u' = u + \Delta u$, $\Delta u \to 0$ transforms under the transformation $u^{\mu} = u^{\mu}(\tilde{u})$ of the parameters' space. As a consequence, we shall derive a transformation rule for the Christoffel symbols related with the connection $\bar{\varphi}$ of *S*.

According to the general properties of a connection, we have:

$$\bar{\varphi}_{u+\Delta u,u}\left(\bar{\xi}(u)\right) = \bar{\xi}_{\bar{\varphi}}(u+\Delta u) \in T_{u+\Delta u}S$$

$$e_{\nu}(u+\Delta u)\bar{\varphi}_{\mu}^{\nu}(u+\Delta u,u)\bar{\xi}^{\mu}(u) = e_{\nu}(u+\Delta u)\bar{\xi}_{\bar{\varphi}}{}^{\nu}(u+\Delta u)$$

$$\bar{\varphi}_{\mu}^{\nu}(u+\Delta u,u)\bar{\xi}^{\mu}(u) = \bar{\xi}_{\bar{\varphi}}{}^{\nu}(u+\Delta u)$$

$$\bar{\varphi}_{\mu}^{\nu}(u+\Delta u,u)J_{\kappa}^{\mu}(\tilde{u})\tilde{\bar{\xi}}^{\kappa}(\tilde{u}) = J_{\lambda}^{\nu}(\tilde{u}+\Delta\tilde{u})\tilde{\bar{\xi}}_{\bar{\varphi}}{}^{\lambda}(\tilde{u}+\Delta\tilde{u})$$

By definition:

$$\tilde{u}^{\mu} + \Delta\tilde{u}^{\mu} = \tilde{u}^{\mu}(u+\Delta u) \approx \tilde{u}^{\mu} + \tilde{J}_{\nu}^{\mu}(u)\Delta u^{\nu}$$

Given that the matrix $\left[\tilde{J}_{\lambda}^{\nu}(u+\Delta u)\right]$ is the inverse of $\left[J_{\lambda}^{\nu}(\tilde{u}+\Delta\tilde{u})\right]$ we obtain:

$$\tilde{J}_{\nu}^{\lambda}(u+\Delta u)\bar{\varphi}_{\mu}^{\nu}(u+\Delta u,u)J_{\kappa}^{\mu}(\tilde{u})\tilde{\bar{\xi}}^{\kappa}(\tilde{u}) = \tilde{\bar{\xi}}_{\bar{\varphi}}{}^{\lambda}(\tilde{u}+\Delta\tilde{u})$$

In the system of parameters $\tilde{u} = (\tilde{u}^1, \tilde{u}^2)$ the matrix elements of $\bar{\varphi}$ satisfy the relation:

$$\tilde{\bar{\xi}}_{\bar{\varphi}}{}^{\lambda}(\tilde{u}+\Delta\tilde{u}) = \tilde{\bar{\varphi}}_{\kappa}^{\lambda}(\tilde{u}+\Delta\tilde{u},\tilde{u})\tilde{\bar{\xi}}^{\kappa}(\tilde{u})$$

Hence, by combining the last two relationships, we imply that:

$$\tilde{\bar{\varphi}}_{\kappa}^{\lambda}(\tilde{u}+\Delta\tilde{u},\tilde{u}) = \tilde{J}_{\nu}^{\lambda}(u+\Delta u)\bar{\varphi}_{\mu}^{\nu}(u+\Delta u,u)J_{\kappa}^{\mu}(\tilde{u})$$

We expand in Taylor series and keep terms up to the first order with respect to the infinitesimal quantities:

$$\delta_{\kappa}^{\lambda} + \tilde{\Gamma}_{\kappa\rho}^{\lambda}(\tilde{u})\Delta\tilde{u}^{\rho} = \left(\tilde{J}_{\nu}^{\lambda}(u) + \frac{\partial\tilde{J}_{\nu}^{\lambda}(u)}{\partial u^{\rho}}\Delta u^{\rho}\right)\left(\delta_{\mu}^{\nu} + \Gamma_{\mu\sigma}^{\nu}(u)\Delta u^{\sigma}\right)J_{\kappa}^{\mu}(\tilde{u})$$

$$\tilde{\Gamma}_{\kappa\rho}^{\lambda}(\tilde{u})\tilde{J}_{\sigma}^{\rho}(u)\Delta u^{\sigma} = \left(\Gamma_{\mu\sigma}^{\nu}(u)\tilde{J}_{\nu}^{\lambda}(u)J_{\kappa}^{\mu}(\tilde{u}) + \frac{\partial\tilde{J}_{\nu}^{\lambda}(u)}{\partial u^{\sigma}}J_{\kappa}^{\nu}(\tilde{u})\right)\Delta u^{\sigma}$$

Finally, we obtain the identity:

$$\tilde{\Gamma}_{\kappa\rho}^{\lambda}(\tilde{u}) = \Gamma_{\mu\sigma}^{\nu}(u)\tilde{J}_{\nu}^{\lambda}(u)J_{\kappa}^{\mu}(\tilde{u})J_{\rho}^{\sigma}(\tilde{u}) + \frac{\partial\tilde{J}_{\nu}^{\lambda}(u)}{\partial u^{\sigma}}J_{\kappa}^{\nu}(\tilde{u})J_{\rho}^{\sigma}(\tilde{u}) \qquad (11.12)$$

Remark: According to 11.12 the Christoffel symbols do **not** transform like a tensor field (see paragraph 9: Transformation of the metric tensor). In addition, we notice that the second term on the right-hand-side of 11.12 does not depend at all on the Christoffel symbols. It is an expression of the specific parameter-transformation. We deduce that it is possible the Christoffel symbols be identically zero in one system of parameters, but different of zero in another. In a Euclidean plane, it is possible to choose the parameters so that the corresponding Christoffel symbols are not all zero; for example in polar coordinates: see Examples 11 A and B, in the present paragraph.

Connections which are symmetric and compatible with the metric tensor of the geometric surface

In the example 11A, we saw that it is possible to define many different connections in a geometric surface. From now on, we restrict our interest on connections that satisfy two specific conditions:

a) They are "symmetric".

b) They are "compatible" with the metric tensor of the geometric surface.
Let us see what these restrictions mean and how they help us to evaluate the Christoffel symbols corresponding to connections of this kind.

Remark: The Christoffel symbols carry significant information about the behavior and the properties of a geometric surface: we have already seen that they determine the way a vector field change along a curve of the geometric surface; we shall find out that they also play a role of crucial importance for the determination of the curves with minimal length joining two points of the plane and for the definition of the "curvature" of the geometric surface.

Symmetric connections:
We say that a connection $\bar{\varphi}$ is "symmetric", if for every $u=\left(u^1,u^2\right)\in B\subseteq \boldsymbol{R}^2$ the corresponding Christoffel symbols satisfy the relationship:

$$\bar{\Gamma}^{\mu}_{\nu\kappa}(u)=\bar{\Gamma}^{\mu}_{\kappa\nu}(u) \tag{11.13}$$

Connections compatible with the metric tensor:
We say that $\bar{\varphi}$ is compatible with the metric tensor of the geometric surface *S*, if every isomorphism $\bar{\varphi}^{a}_{u'u}:T_uS\to T_{u'}S$ is an isometry:
Consider any two vectors $\bar{\xi}$, $\bar{\zeta}\in T_uS$ then, if $\bar{\varphi}$ is compatible with the metric tensor of the geometric surface *S* then, for any curve *a* of the parameters' space joining *u* and *u'*, the inner product of $\bar{\xi}$, $\bar{\zeta}\in T_uS$ has the same value with the inner product of their images under $\bar{\varphi}^{a}_{u'u}$ i.e.:

$$\left\langle \bar{\xi}(u),\bar{\zeta}(u)\right\rangle=\left\langle \bar{\varphi}^{a}_{u',u}\left(\bar{\xi}(u)\right),\bar{\varphi}^{a}_{u',u}\left(\bar{\zeta}(u)\right)\right\rangle \tag{11.14}$$

If a connection $\bar{\varphi}$ satisfies the previous assumptions (a) and (b), we are able to calculate the corresponding Christoffel symbols as functions of the matrix elements $g_{\mu\nu}(u)$ and their first order partial derivatives $\partial_\lambda g_{\mu\nu}(u)$ of the metric tensor, as follows:
Let $u'=u+\Delta u$, $\Delta u=\left(\Delta u^1,\Delta u^2\right)\to(0,0)$ be point of the parameters' space infinitesimally close to *u*. We begin with (11.14) and obtain sequentially the relationships (in the subsequent equations we expand the functions with argument $u+\Delta u$ in Taylor series and we keep terms up to the first order with respect to the infinitesimal quantities):

$$\left\langle \bar{\xi}(u),\bar{\zeta}(u)\right\rangle=\left\langle \bar{\varphi}^{a}_{u+\Delta u,u}\left(\bar{\xi}(u)\right),\bar{\varphi}^{a}_{u+\Delta u,u}\left(\bar{\zeta}(u)\right)\right\rangle$$

$$\left\langle e_\mu(u),e_\nu(u)\right\rangle\bar{\xi}^{\mu}(u)\bar{\zeta}^{\nu}(u)=\left\langle \bar{\varphi}^{a}_{u+\Delta u,u}\left(e_\mu(u)\right),\bar{\varphi}^{a}_{u+\Delta u,u}\left(e_\nu(u)\right)\right\rangle\bar{\xi}^{\mu}(u)\bar{\zeta}^{\nu}(u)$$

$$g_{\mu\nu}(u)=\left\langle e_\kappa(u+\Delta u),e_\lambda(u+\Delta u)\right\rangle\bar{\varphi}^{\kappa}_{\mu}(u+\Delta u,u)\bar{\varphi}^{\lambda}_{\nu}(u+\Delta u,u)$$

$$g_{\mu\nu}(u)=g_{\kappa\lambda}(u+\Delta u)\bar{\varphi}^{\kappa}_{\mu}(u+\Delta u,u)\bar{\varphi}^{\lambda}_{\nu}(u+\Delta u,u)$$

$$g_{\mu\nu}=\left(g_{\kappa\lambda}+\partial_\rho g_{\kappa\lambda}\Delta u^\rho\right)\left(\delta^{\kappa}_{\mu}+\Gamma^{\kappa}_{\mu\sigma}\Delta u^\sigma\right)\left(\delta^{\lambda}_{\nu}+\Gamma^{\lambda}_{\nu\tau}\Delta u^\tau\right)$$

If not written explicitly, the argument of the functions is *u*. The metric tensor is symmetric; hence, we have

$$\left(g_{\mu\lambda}\Gamma^{\lambda}_{\nu\rho}+g_{\kappa\nu}\Gamma^{\kappa}_{\mu\rho}+\partial_\rho g_{\mu\nu}\right)\Delta u^\rho=0$$

According to 11.11a relating the different forms of the Christoffel symbols, we result the equations:

$$g_{\mu\lambda}\bar{\Gamma}^{\lambda}_{\nu\rho}+g_{\nu\kappa}\bar{\Gamma}^{\kappa}_{\mu\rho}-\partial_\rho g_{\mu\nu}=0$$

$g_{\mu\lambda}\bar{\Gamma}^{\lambda}_{\nu\rho} + g_{\kappa\nu}\bar{\Gamma}^{\kappa}_{\mu\rho} - \partial_\rho g_{\mu\nu} = 0$

$$\bar{\Gamma}_{\mu\nu\rho} + \bar{\Gamma}_{\nu\mu\rho} = \partial_\rho g_{\mu\nu} \tag{11.15a}$$

By cyclic permutation of the indices, from (10.15a) we obtain the additional equations:

$$\bar{\Gamma}_{\nu\rho\mu} + \bar{\Gamma}_{\rho\nu\mu} = \partial_\mu g_{\nu\rho} \tag{11.15b}$$

$$\bar{\Gamma}_{\rho\mu\nu} + \bar{\Gamma}_{\mu\rho\nu} = \partial_\nu g_{\rho\mu} \tag{11.15c}$$

According to our assumption (a), the Christoffel symbols are symmetric; hence from 11.15.a-c we obtain that:

$$\bar{\Gamma}_{\mu\nu\rho} + \bar{\Gamma}_{\rho\mu\nu} + \bar{\Gamma}_{\nu\rho\mu} = \frac{1}{2}\left(\partial_\mu g_{\nu\rho} + \partial_\rho g_{\mu\nu} + \partial_\nu g_{\rho\mu}\right) \tag{11.15d}$$

Combining 11.15d with each one of the relations 11.15a-c, we deduce that the Christoffel symbols are calculated by the relationship:

$$\bar{\Gamma}_{\mu\nu\rho} = \frac{1}{2}\left(-\partial_\mu g_{\nu\rho} + \partial_\rho g_{\mu\nu} + \partial_\nu g_{\rho\mu}\right) \tag{11.16}$$

Parallel displacement and covariant differentiation of a vector field

We have already seen that a vector field of any geometric surface, generated by the parallel transport of a vector along a curve of its parameters' space, is determined by the equation:

$$\xi(t) = \varphi^{(a)}_{t,t_0}\left(\xi_{(0)}\right) \tag{11.17}$$

In 11.17 φ is any connection defined on the geometric surface *S*.

In this section we are going to confirm that the covariant derivative of a vector field $\xi(t) = \bar{\xi}\left(a(t)\right)$ transported parallel to itself along the curve *a*, is zero:

$$\frac{D^{\mu}_{(a)}\xi(t)}{Dt} = \frac{d\xi^{\mu}(t)}{dt} + \bar{\Gamma}^{\mu}_{\nu\kappa}\left(a(t)\right)\xi^{\nu}(t)\frac{da^{\kappa}(t)}{dt} = 0 \tag{11.18a}$$

$$\xi(t_0) = \bar{\xi}\left(a(0)\right) \underset{def}{=} \xi_{(0)} \tag{11.18b}$$

Or, in other words, we are going to verify that the vector field determined by 11.17 is a solution of the differential equation 11.18a, with initial condition given by 11.18b.

We start from 11.17; for $\Delta t \to 0$ we have:

$\xi(t+\Delta t) = \varphi^{(a)}_{t+\Delta t,t_0}\left(\xi(t_0)\right) = \varphi^{(a)}_{t+\Delta t,t}\left(\varphi^{(a)}_{t,t_0}\left(\xi(t_0)\right)\right) = \varphi^{(a)}_{t+\Delta t,t}\left(\xi(t)\right)$

Symbolize: $\bar{\varphi}^{(a)}_{a(t'),a(t)}\left(\xi(t)\right) \underset{def}{=} \varphi^{(a)}_{t',t}\left(\xi(t)\right)$

$\xi(t+\Delta t) = \varphi^{(a)}_{t+\Delta t,t}\left(\xi(t)\right) \underset{def}{=} \bar{\varphi}^{(a)}_{a(t+\Delta t),a(t)}\left(\xi(t)\right) = \bar{\varphi}^{(a)}_{a(t+\Delta t),a(t)}\left(e_\nu(t)\xi^{\nu}(t)\right)$

$e_\mu\left(a(t+\Delta t)\right)\xi^{\mu}(t+\Delta t) = \bar{\varphi}^{(a)}_{a(t+\Delta t),a(t)}\left(e_\nu(t)\right)\xi^{\nu}(t)$

$e_\mu\left(a(t+\Delta t)\right)\xi^{\mu}(t+\Delta t) = e_\mu\left(a(t+\Delta t)\right)\bar{\varphi}^{(a)\mu}_{\nu}\left(a(t+\Delta t), a(t)\right)\xi^{\nu}(t)$

As usually, we reject terms of order greater than one:

$$\xi^{\mu}(t+\Delta t) = \left(\delta^{\mu}_{\nu} + \left.\frac{\partial\bar{\varphi}^{(a)\mu}_{\nu}\left(u, a(t)\right)}{\partial u^{\kappa}}\right|_{u=a(t)}\frac{da^{\kappa}(t)}{dt}\Delta t\right)\xi^{\nu}(t)\ ,\ \Delta t \to 0$$

$$\frac{d\xi^{\mu}(t)}{dt} = \Gamma^{\mu}_{\nu\kappa}\left(a(t)\right)\xi^{\nu}(t)\frac{da^{\kappa}(t)}{dt} = -\bar{\Gamma}^{\mu}_{\nu\kappa}\left(a(t)\right)\xi^{\nu}(t)\frac{da^{\kappa}(t)}{dt}$$ (See 11.11a)

$$\frac{d\xi^{\mu}(t)}{dt} + \bar{\Gamma}^{\mu}_{\nu\kappa}\left(a(t)\right)\xi^{\nu}(t)\frac{da^{\kappa}(t)}{dt} = 0$$

Furthermore $\xi(t)$ given by 11.17, satisfies the initial condition:

$\xi\left(t_{t_0}\right) = \varphi^{(a)}_{t_0,t_0}\left(\xi_{(0)}\right) = \xi_{(0)}$.

Remarks:

1) Let φ be the connection defined on the geometric surface *S*. We presume that for some system of parameters $u=(u^1,u^2)$ the corresponding Christoffel symbols are identically zero in every tangent space of *S*. Then from 11.18 we deduce that the components of the vector field $\xi_{(\varphi)}(t)$ generated by the parallel transport of any vector $\xi_{(0)} \in T_{u_{(0)}}S$ (including the basis-vectors) along any curve *a* of the parameters' space passing by $u_{(0)}$ satisfy the equations:

$$\frac{d\xi_{(\varphi)}{}^{\mu}(t)}{dt} = 0$$

I.e. the coordinates of $\xi_{(\varphi)}(t)$ are constant and equal to the initial ones:

$$\xi_{(\varphi)}{}^{\mu}(t) = \xi_{(0)}{}^{\mu}$$

This is the well-known case of the Euclidean plane, where the Christoffel symbols of the connection which is compatible with the Euclidean metric are identically zero (see 11.16).

2) **Infinitesimal variation of the basis-vectors fields $e_\mu(u) \in T_uS$ with respect to the connection $\bar{\varphi}$ defined on *S***

We calculate the covariant differentials of the basis-vector fields $e_\mu(u) \in T_uS$ with respect to the connection $\bar{\varphi}$ defined on *S*:

$$D_{\Delta u}e_\mu(u) = \bar{\varphi}_{u,u+\Delta u}\left(e_\mu(u+\Delta u)\right) - e_\mu(u) = e_\nu(u)\bar{\varphi}^\nu_\mu(u,u+\Delta u) - e_\nu(u)\delta^\nu_\mu =$$

$$= e_\nu(u)\left(\delta^\nu_\mu + \left.\frac{\partial\bar{\varphi}^\nu_\mu(u,\mathrm{v})}{\partial v^\kappa}\right|_{v=u}\Delta u^\kappa - \delta^\nu_\mu\right) = e_\nu(u)\bar{\Gamma}^\nu_{\mu\kappa}(u)\Delta u^\kappa$$

$$D_{\Delta u}e_\mu(u) = e_\nu(u)\bar{\Gamma}^\nu_{\mu\kappa}(u)\Delta u^\kappa \qquad (11.19)$$

3) Assume that the connection φ is symmetric and compatible with the metric tensor; this implies that φ is an isometry (relation 11.14); hence, the norm of the parallel displaced field $\bar{\xi}(a(t)) = \varphi^{(a)}_{t,t_0}\left(\bar{\xi}_{(0)}\right)$ is constant:

$$\frac{d\left|\bar{\xi}(a(t))\right|}{dt} = 0$$

4) **Infinitesimal change of a real function defined on a vector field of the geometric surface S**

We assume that the points of the geometric surface *S* are determined by the system of parameters $u=(u^1,u^2)$: $\boldsymbol{R}^2 \supseteq B \ni u = \left(u^1,\mathrm{u}^2\right) \leftrightarrow P_u \in S$

Consider a vector field $\bar{\xi}(u)$ on *S*. The range of the vector field is the set $\bar{\xi}(B)$ which is a subset of the union of the tangent spaces of *S*:

$$\bar{\xi}(B) \subseteq \bigcup_{u\in B} T_uS$$

The norm of the vector field $\bar{\xi}(u)$ is an example of a continuous real function defined on the subset the union of the tangent space of *S*:

$$\left|\bar{\xi}(u)\right| = \sqrt{\left\langle \bar{\xi}(u), \bar{\xi}(u) \right\rangle} = \sqrt{g_{\mu\nu}(u)\bar{\xi}^{\mu}(u)\bar{\xi}^{\nu}(u)}$$

$$\bigcup_{u\in B} T_u S \supseteq \xi(B) \overset{\|\bar{\xi}\|}{\to} \boldsymbol{R}$$

The calculation of the covariant differential of $\left|\bar{\xi}(u)\right|$ is achieved by applying the properties of the infinitesimal covariant variation of the field $\bar{\xi}(u)$ with respect to the connection defined on the geometric surface. In general, assume a real differentiable function *F* with domain the vector fields defined on *S*. The covariant differential of *F* at the point P_u in *S* along the infinitesimal change of parameters $\Delta u = \left(\Delta u^1, \Delta u^2\right)$ is defined by the relationship:

$$\begin{aligned} D_{\Delta u}F\left(\bar{\xi}(u)\right) &= \left[F\left(\bar{\xi}(u) + D_{\Delta u}\bar{\xi}(u)\right) - F\left(\bar{\xi}(u)\right)\right]_{\Delta u\to 0} \approx \frac{\partial F\left(\bar{\xi}\right)}{\partial \xi^{\mu}} D^{\mu}_{\Delta u}\bar{\xi}(u) = \\ &= \frac{\partial F\left(\bar{\xi}\right)}{\partial \xi^{\mu}}\left(\partial_{\kappa}\bar{\xi}^{\mu} + \bar{\Gamma}^{\mu}_{\nu\kappa}\bar{\xi}^{\nu}\right)\Delta u^{\kappa} \end{aligned} \tag{11.20}$$

We apply 11.20 for the matrix elements of the metric tensor and we check the results, comparing them with the relationships obtained in the previous sections of the present paragraph (see relations 11.15). We have:

$$D_{\Delta u}g_{\mu\nu}(u) = D_{\Delta u}\left\langle e_{\mu}(u), e_{\nu}(u)\right\rangle = \left\langle e_{\mu}(u) + D_{\Delta u}e_{\mu}(u), e_{\nu}(u) + D_{\Delta u}e_{\nu}(u)\right\rangle - \left\langle e_{\mu}(u), e_{\nu}(u)\right\rangle \approx$$

$$\approx g_{\mu\lambda}(u)\bar{\Gamma}^{\lambda}_{\nu\kappa}(u)\Delta u^{\kappa} + g_{\nu\lambda}(u)\bar{\Gamma}^{\lambda}_{\mu\kappa}(u)\Delta u^{\kappa} = \left(\bar{\Gamma}_{\mu\nu\kappa} + \bar{\Gamma}_{\nu\mu\kappa}\right)\Delta u^{\kappa}$$

On the other hand it holds that:

$$D_{\Delta u}g_{\mu\nu}(u) = \left\langle \varphi_{u,u+\Delta u}\left(e_{\mu}(u+\Delta u)\right), \varphi_{u,u+\Delta u}\left(e_{\nu}(u+\Delta u)\right)\right\rangle - \left\langle e_{\mu}(u), e_{\nu}(u)\right\rangle$$

But φ is an isometry; hence we obtain:

$$D_{\Delta u}g_{\mu\nu}(u) = \left\langle e_{\mu}(u+\Delta u), e_{\nu}(u+\Delta u)\right\rangle - \left\langle e_{\mu}(u), e_{\nu}(u)\right\rangle = g_{\mu\nu}(u+\Delta u) - g_{\mu\nu}(u) \approx \partial_{\kappa}g_{\mu\nu}(u)\Delta u^{\kappa}$$

We combine the above relations and we result the well-known equation 11.15:

$$\bar{\Gamma}_{\mu\nu\kappa} + \bar{\Gamma}_{\nu\mu\kappa} = \partial_{\kappa}g_{\mu\nu}$$

5) Verify the following identities: [We symbolize *F* any a real function defined on the vector fields of *S* - $\bar{\xi}(u), \bar{\zeta}(u) \in T_u S$, $\Delta_{(j)}u = \left(\Delta_{(j)}u^1, \Delta_{(j)}u^2\right)$ $j = 1,2,3...$ and $\lambda \in \boldsymbol{R}$]

5A) $D_{\Delta_{(1)}u+\Delta_{(2)}u}\bar{\xi}(u) = D_{\Delta_{(1)}u}\bar{\xi}(u) + D_{\Delta_{(2)}u}\bar{\xi}(u)$

5B) $D_{\Delta u}\left(\bar{\xi}(u) + \bar{\zeta}(u)\right) = D_{\Delta u}\bar{\xi}(u) + D_{\Delta u}\bar{\zeta}(u)$

5C) $D_{\Delta u}\left(\lambda\bar{\xi}(u)\right) = \lambda D_{\Delta u}\left(\bar{\xi}(u)\right)$

5D) $D_{\lambda\Delta u}\bar{\xi}(u) = \lambda D_{\Delta u}\bar{\xi}(u)$

5E) $D_{\Delta u}\left(\bar{\xi}(u)F\left(\bar{\zeta}(u)\right)\right) = D_{\Delta u}\bar{\xi}(u)F\left(\bar{\zeta}(u)\right) + \bar{\xi}(u)D_{\Delta u}\left(F\left(\bar{\zeta}(u)\right)\right)$

5F) $D_{\Delta u}\left\langle \bar{\xi}(u), \bar{\zeta}(u)\right\rangle = \left\langle D_{\Delta u}\bar{\xi}(u), \bar{\zeta}(u)\right\rangle + \left\langle \bar{\xi}(u), D_{\Delta u}\bar{\zeta}(u)\right\rangle$

Example 11B

The Euclidean Plane in Polar Coordinates - Connection - Covariant Differentiation - Parallel Transport

In this example, we describe the Euclidean plane in polar coordinates:

a) We find the basis-elements at every tangent space as linear combinations of the natural basis
b) We derive the explicit form of the metric tensor in polar coordinates
c) We calculate the Christoffel symbols in polar coordinates
d) We find the analytic expression of a parallel transported field along any curve of the Euclidean plane, in polar coordinates
e) We confirm that the geometric features of the Euclidean plane, like the properties of a parallel transported field are not affected by the coordinate transformation

Consider the Euclidean plane E and a Cartesian coordinates system $x=(x^1,x^2)$ in it. In every tangent space T_xE of E, the metric tensor $g(x)$ is identical to the unit matrix:

$$\left[g_{\mu\nu}\right]=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

The basis-vectors $\mathbf{x}_\mu \ \mu=1,2$ of every tangent space T_xE satisfy the relationships:

$$g_{\mu\nu}=\left\langle \mathbf{x}_\mu,\mathbf{x}_\nu\right\rangle=\delta_{\mu\nu}$$

The norm of any vector $\Delta x=\mathbf{x}_\mu\Delta x^\mu\in T_xE$ is calculated by the formula:

$$\left|\Delta x\right|=\sqrt{g_{\mu\nu}\Delta x^\mu\Delta x^\nu}=\sqrt{\left(\Delta x^1\right)^2+\left(\Delta x^2\right)^2}$$

We apply a coordinate-transformation on E, from the Cartesian to polar coordinates:

$$\begin{aligned}x^1&=u^1\cos\left(u^2\right)\\ x^2&=u^1\sin\left(u^2\right)\end{aligned}\qquad\text{(E11B.1)}$$

The induced transformation on the tangent spaces of E is determined by the relations:

$$\Delta x^\mu=\frac{\partial x^\mu}{\partial u^\nu}\Delta u^\nu\ ,\ \Delta x=\mathbf{x}_\mu\Delta x^\mu\in T_xE$$

According to E11B.1:

$$\begin{aligned}\Delta x^1&=\Delta u^1\cos u^2-\Delta u^2u^1\sin u^2\\ \Delta x^2&=\Delta u^1\sin u^2+\Delta u^2u^1\cos u^2\end{aligned}\qquad\text{(E11B.2)}$$

The basis-vectors $e_\mu(u)\ \mu=1,2$ corresponding to the polar coordinate-system are calculated by the relationships:

$$\Delta x=\mathbf{x}_\mu\Delta x^\mu=e_\nu(u)\,\Delta u^\nu$$

$$e_\nu(u)=\mathbf{x}_\mu\frac{\partial x^\mu}{\partial u^\nu}$$

$$\begin{aligned}e_1(u)&=\mathbf{x}_1\cos u^2+\mathbf{x}_2\sin u^2\\ e_2(u)&=-\mathbf{x}_1u^1\sin u^2+\mathbf{x}_2u^1\cos u^2\end{aligned}\qquad\text{(E11B.3)}$$

From E11B.3, we obtain the form of the metric tensor in the u-parameters:

$$\bar g_{\mu\nu}(u)=\left\langle e_\mu(u),e_\nu(u)\right\rangle$$

$$\left[\bar g_{\mu\nu}(u)\right]=\begin{pmatrix}1 & 0\\ 0 & \left(u^1\right)^2\end{pmatrix}\qquad\text{(E11B.4)}$$

Now, make an abstraction and imagine that the plane E as a geometric surface whose points are determined by the couples:

$$u=\left(u^1,u^2\right)$$

$$u^1\in(0,+\infty)\ \ u^2\in\bigcup_{k=-\infty}^{+\infty}\left[2\pi(k-1),2\pi k\right)\ \ k\in\mathbf{Z}$$

At every tangent space T_uE of E, the metric tensor with respect to the basis $\{e_1(u),e_2(u)\}$, is given by E11B.4. The norm of any vector $\Delta U = e_\nu(u)\Delta u^\nu \in T_uE$ is calculated by the relationship:

$$|\Delta U| = \left(\left(\Delta u^1\right)^2 + \left(u^1\right)^2\left(\Delta u^1\right)^2\right)^{1/2}$$

What about the connection $\bar{\varphi}$ in E which is compatible with the metric tensor E11B.4?

To determine the connection, it is enough to calculate the corresponding Christoffel symbols, according to 11.16. After some tedious calculations we find that:

$$\begin{aligned} &\bar{\Gamma}_{111} = \bar{\Gamma}_{112} = \bar{\Gamma}_{121} = \bar{\Gamma}_{211} = \bar{\Gamma}_{222} = 0 \\ &\bar{\Gamma}_{122} = -u^1 \\ &\bar{\Gamma}_{212} = \bar{\Gamma}_{221} = u^1 \end{aligned} \qquad \text{(E11B.5)}$$

The symbols $\bar{\Gamma}^\nu_{\mu\kappa}(u)$ are determined by the identities:

$$\bar{\Gamma}^\nu_{\mu\kappa}(u) = \bar{g}^{\nu\lambda}(u)\bar{\Gamma}_{\lambda\mu\kappa}(u)$$

The matrix $\bar{g}(u) = \left[\bar{g}^{\nu\lambda}(u)\right]$ is the inverse of E11B.4:

$$\left[\bar{g}^{\nu\lambda}(u)\right] = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{\left(u^1\right)^2} \end{pmatrix}$$

We obtain:

$$\begin{aligned} &\bar{\Gamma}^1_{11} = \bar{\Gamma}^1_{12} = \bar{\Gamma}^1_{21} = \bar{\Gamma}^2_{11} = \bar{\Gamma}^2_{22} = 0 \\ &\bar{\Gamma}^1_{22} = -u^1 \\ &\bar{\Gamma}^2_{12} = \bar{\Gamma}^2_{21} = \frac{1}{u^1} \end{aligned} \qquad \text{(E11B.5)}$$

Let us now examine how a vector $\xi_{(0)} \in T_{u_{(0)}}E$ is transported parallel to itself along a curve $a: u^\mu = a^\mu(t)$ of the parameters' space, in the u-parameter-system.

We apply equation 11.18 and we obtain the system of differential equations:

$$\frac{d\xi^1}{dt} - a^1\dot{a}^2\xi^2 = 0 \qquad \text{(E11B.6a)}$$

$$\frac{d\xi^2}{dt} + \frac{\dot{a}^2}{a^1}\xi^1 + \frac{\dot{a}^1}{a^1}\xi^2 = 0 \qquad \text{(E11B.6b)}$$

We change the parameterization of the curve, so that:

$t = a^2$, $a^1 = a^1\left(a^2\right)$

Equations E11B.6a and b take the subsequent forms:

$$\frac{d\xi^1}{da^2} - a^1\xi^2 = 0$$

$$a^1\frac{d\xi^2}{da^2} + \frac{da^1}{da^2}\xi^2 + \xi^1 = 0$$

$$\frac{d\xi^1}{da^2} - a^1\xi^2 = 0$$

$$\frac{d}{da^2}\left(a^1\xi^2\right) + \xi^1 = 0$$

By setting:

$$\zeta^1 \underset{def}{=} \xi^1 \quad \zeta^2 \underset{def}{=} a^1\xi^2 \tag{E11B.7}$$

The system of the differential equations is transformed as follows:

$$\begin{aligned} \frac{d\zeta^1}{da^2} - \zeta^2 &= 0 \\ \frac{d\zeta^2}{da^2} + \zeta^1 &= 0 \end{aligned} \tag{E11B.8}$$

The system of equations E11B.7 and 8 is solvable for any curve *a* of the plane *E*, under the constrain $a^1\left(a^2\right) \neq 0$ and the initial conditions:

$\xi^1\left(a_{(0)}{}^2\right) = \xi_{(0)}{}^1$, $\xi^2\left(a_{(0)}{}^2\right) = \xi_{(0)}{}^2$, $a_{(0)}{}^1 = a^1\left(a_{(0)}{}^2\right)$

We find that:

$$\begin{aligned} \xi^1 &= A\sin\left(a^2 + \theta\right) \\ \xi^2 &= A\frac{\cos\left(a^2 + \theta\right)}{a^1\left(a^2\right)} \end{aligned} \tag{E11B.9}$$

The constants *A* and θ are determined by the initial conditions:

$$\theta = -a_{(0)}{}^2 + \tan^{-1}\left(\frac{\xi_{(0)}{}^1}{a_{(0)}{}^1 \cdot \xi_{(0)}{}^2}\right)$$

$$A = \sqrt{\left(a_{(0)}{}^1 \cdot \xi_{(0)}{}^2\right)^2 + \left(\xi_{(0)}{}^1\right)^2}$$

Remarks:

A) Given that the connection is symmetric and compatible with the metric of the plane, we expect that the norm of the parallel displaced field $\xi\left(a^2\right)$ is a constant along the curve *a*. Let's check it (see E11B.4):

$$\frac{d\left|\xi(a^2)\right|^2}{da^2} = \frac{d}{da^2}\left(g_{\mu\nu}\xi^\mu\xi^\nu\right) = \frac{d}{da^2}\left(g_{11}(\xi^1)^2 + g_{22}(\xi^2)^2\right) =$$

$$= \frac{d}{da^2}\left(g_{11}(\xi^1)^2 + g_{22}(\xi^2)^2\right) = \frac{d}{da^2}\left(A^2\sin^2\left(a^2 + \theta\right) + \left(a^1\right)^2 A^2\frac{\cos^2\left(a^2 + \theta\right)}{\left(a^1\right)^2}\right) =$$

$$= \frac{d}{da^2}\left(A^2\right) = 0$$

This is obvious and from the equations E11B.6 or 7; it is easily implied that:

$$\frac{d}{dt}\left((\xi^1)^2 + (a^1 \cdot \xi^2)^2\right) = 0$$

$$\frac{d}{dt}\left|\xi(t)\right|^2 = 0$$

B) Under the parameter transformation E11B.1, the Euclidean "nature" of the plane *E* is not affected. That implies that the Cartesian coordinates of the parallel transported field $\xi\left(a^2\right)$ along any curve *a* should be the same for any value of a^2; let's check it:

$$\xi = e_\mu\xi^\mu = \left(\boldsymbol{x_1}\cos a^2 + \boldsymbol{x_2}\sin a^2\right)\xi^1 + \left(\boldsymbol{-x_1}a^1\sin a^2 + \boldsymbol{x_2}a^1\cos a^2\right)\xi^2 =$$

$$= \boldsymbol{x_1}\left(\xi^1\cos a^2 - \xi^2 a^1\sin a^2\right) + \boldsymbol{x_2}\left(\xi^1\sin a^2 + \xi^2 a^1\cos a^2\right) =$$

$$= \boldsymbol{x_1}A\left(\sin(a^2+\theta)\cos a^2 - \cos(a^2+\theta)\sin a^2\right) + \boldsymbol{x_2}A\left(\sin(a^2+\theta)\sin a^2 + \cos(a^2+\theta)\cos a^2\right) =$$

$$= \boldsymbol{x_1}A\sin\theta + \boldsymbol{x}_2 A\cos\theta$$

C) Let us consider a closed curve *a* of *E*. Say that $a^1(2\pi) = a^1(0)$ (see relations E11B.1). We pick a vector $\xi_{(0)} \in T_{a(o)}E$ and we transport it along *a*, until we return at the initial tangent space: $T_{a(2\pi)}E = T_{a(o)}E$

Name $\xi_{(1)}$ the image of $\xi_{(0)}$ at the end of its trip:

$$\xi_{(1)} = \varphi^{(a)}_{2\pi,0}\left(\xi_{(0)}\right) \in T_{a(2\pi)}E$$

We anticipate that for the Euclidean plane the initial vector $\xi_{(0)}$ and its image $\xi_{(1)}$ are identical: $\xi_{(1)} = \xi_{(0)}$

We are going to check that this is true. [We notice that as we shall see in the next paragraphs, this is not the general case for a geometric surface]

We have:

$$\xi^1\left(a^2\right) = A\sin\left(a^2 + \theta\right)$$

$$\xi^2\left(a^2\right) = A\frac{\cos\left(a^2 + \theta\right)}{a^1\left(a^2\right)}$$

Hence:

$$d\xi^1\left(a^2\right) = A\cos\left(a^2 + \theta\right)da^2$$

$$\xi_{(1)}{}^1 - \xi_{(0)}{}^1 = \int_{a^2=0}^{a^2=2\pi} d\xi^1\left(a^2\right) = \int_{a^2=0}^{a^2=2\pi} A\cos\left(a^2 + \theta\right)da^2 = 0$$

We conclude that:

$$\xi_{(1)}{}^1 = \xi_{(0)}{}^1$$

Similarly, for the other coordinate we have:

$$\xi_{(1)}{}^2 - \xi_{(0)}{}^2 = \int_{a^2=0}^{a^2=2\pi} d\xi^2\left(a^2\right) = A\int_{a^2=0}^{a^2=2\pi} da^2\left(-\frac{\sin\left(a^2 + \theta\right)}{a^1} - \frac{\dot{a}^1\cos\left(a^2 + \theta\right)}{\left(a^1\right)^2}\right) =$$

$$= A\frac{\cos\left(a^2 + \theta\right)}{a^1}\Bigg]_{a^2=0}^{a^2=2\pi} - A\int_{a^2=0}^{a^2=2\pi} da^2\left(-\frac{\dot{a}^1\cos\left(a^2 + \theta\right)}{\left(a^1\right)^2}\right) - A\int_{a^2=0}^{a^2=2\pi} da^2\left(\frac{\dot{a}^1\cos\left(a^2 + \theta\right)}{\left(a^1\right)^2}\right) = 0$$

We imply that:

$$\xi_{(1)}{}^2 = \xi_{(0)}{}^2$$

12. **Curvature of a geometric surface**

In the example 11B of the preceding paragraph, we proved that when a vector $\xi_{(0)} \in T_{a(o)}E$ of the Euclidean plane, is transferred parallel to itself along a closed curve *a* to its initial position, it remains unaltered, either using Euclidean or polar coordinates. In this paragraph we deduce that this is not the case for every geometric surface. We shall see that the variation of a vector field transferred parallel to itself along an infinitesimal loop on the geometric surface, depends on a quantity called "**curvature**" that is independent of the system of the parameters. The curvature is determined by the connection defined on the surface. Because of its invariance under any parameter-transformation, the curvature is a geometric characteristic of the surface. If the curvature of a surface is zero everywhere, the surface has the structure of a Euclidean plane.

Consider a geometric surface *S* equipped with the metric tensor $g(u)$ in the u-parameter-system. Let φ be a symmetric connection on *S*, compatible with the metric tensor.

According to paragraph 11, the Christoffel symbols which determine φ are to be calculated by using relations 11.16.

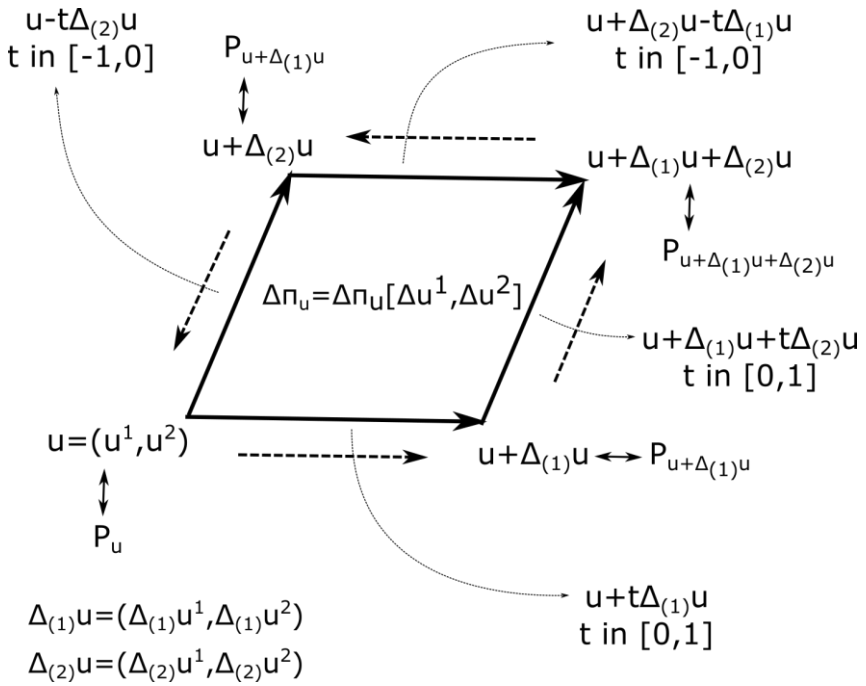

Figure 12.1: An infinitesimal parallelogram $\Delta\pi_u[\Delta_{(1)}u, \Delta_{(2)}u]$ in the u-plane of the parameters is illustrated. Its vertexes u, $u+\Delta_{(1)}u$, $u+\Delta_{(1)}u+\Delta_{(2)}u$, $u+\Delta_{(2)}u$ correspond to the points P_u, $P_{u+\Delta_{(1)}u}$, $P_{u+\Delta_{(1)}u+\Delta_{(2)}u}$, $P_{u+\Delta_{(2)}u}$ of its image $\Delta\Pi_u[\Delta_{(1)}U, \Delta_{(2)}U]$ on S.

The boundary of the infinitesimal parallelogram $\Delta\pi_u[\Delta_{(1)}u, \Delta_{(2)}u]$ is symbolized by $\partial\Delta\pi_u$ and has the analytic expression $u_t=a(t)$; each branch of the curve a is displayed on the figure.

Let $c: u^\mu = c^\mu(t)$ be a closed curve of the domain $B \subseteq \boldsymbol{R}^2$ of the parameters. The region $R \subseteq B : c = \partial R$ enclosed by the closed curve c is possible to be approximated by an infinite number of infinitesimal parallelograms (figure 4.2). The analytic expression of c is possible to be expressed as the summation of the analytic expressions of the boundaries of all these infinitesimal parallelograms. Hence, in order to cope with the main question of this paragraph, it is enough to study the parallel transport of a vector $\xi_{(0)} \in T_uS$ along the boundary of an infinitesimal parallelogram of the u-plane of the parameters (figure 12.1).
Let $a: u_t=a(t)$ be the analytic expression of the boundary $\partial\Delta\pi_u$ of the infinitesimal parallelogram, with respect to a parameter t.
Pick any vector $\xi_{(0)} \in T_uS$, $u = u_0 = a(0)$ and define the vector field:

$$\xi(t) \underset{def}{=} \bar{\xi}\left(a(t)\right) = \bar{\varphi}^{(a)}_{a(t),u}\left(\xi_{(0)}\right)$$

The vector field $\xi(t)$ is the parallel transport of $\xi_{(0)} \in T_uS$ along the closed curve a; the covariant derivative of $\xi(t)$ is zero (relation 11.9b):

$$D^\mu_{(a)}\xi = d\xi^\mu + \bar{\Gamma}^\mu_{\nu\kappa}\left(a(t)\right)\xi^\nu da^\kappa = 0$$

$$d\xi^\mu = -\bar{\Gamma}^\mu_{\nu\kappa}\left(a\right)\xi^\nu\, da^\kappa \qquad (12.1)$$

We integrate 12.1 along the closed curve *a*; we begin from the vertex *u*, we move along the boundary of the infinitesimal parallelogram and return at the initial vertex *u*. Let us name $\xi_{(1)}$ the final vector of the field, when we have returned to the initial tangent space:

$\xi_{(1)} \in T_u S$

We write:

$\xi_{(0)} = e_\mu(u)\xi_{(0)}{}^\mu \in T_u S$

$\xi_{(1)} = e_\mu(u)\xi_{(1)}{}^\mu \in T_u S$

Proceeding with the integration of 12.1, we find that:

$$\xi_{(1)}{}^\mu - \xi_{(0)}{}^\mu = -\oint_{\partial \Delta\Pi_u} \bar{\Gamma}^\mu_{\nu\kappa}(a)\xi^\nu da^\kappa \tag{12.2}$$

We carry out the integration on the right hand of 12.2, by taking into account 12.1 and keeping terms up to the second order with respect to the infinitesimal quantities $\Delta_{(1)}u^\mu$, $\Delta_{(2)}u^\mu$ (figure 12.1). We result the following equation, where the argument of every function is considered at *u*.

$$\xi_{(1)}{}^\mu - \xi_{(0)}{}^\mu = R^\mu_{\nu\lambda\kappa}\xi_{(0)}{}^\nu\left(\Delta_{(1)}u^\lambda\Delta_{(2)}u^\kappa - \Delta_{(1)}u^\kappa\Delta_{(2)}u^\lambda\right) \tag{12.3}$$

The quantities $R^\mu_{\nu\lambda\kappa}$ are defined by the relationships:

$$R^\mu_{\nu\lambda\kappa} \underset{def}{=} \bar{\Gamma}^\mu_{\rho\kappa}\bar{\Gamma}^\rho_{\nu\lambda} - \partial_\lambda\bar{\Gamma}^\mu_{\nu\kappa} \tag{12.4}$$

From 12.3 it is clear that under any parameter transformation $u^\mu = u^\mu(\tilde{u})$ $R^\mu_{\nu\lambda\kappa}$ transforms like a tensor; we name $R^\mu_{\nu\lambda\kappa}$ "**the curvature tensor**" of the geometric surface at its point determined by *u*.

Given that $\kappa,\lambda = 1,2$ 12.3 is equivalently written:

$$\xi_{(1)}{}^\mu - \xi_{(0)}{}^\mu = \left(R^\mu_{\nu 12} - R^\mu_{\nu 21}\right)\xi_{(0)}{}^\nu\left(\Delta_{(1)}u^1\Delta_{(2)}u^2 - \Delta_{(1)}u^2\Delta_{(2)}u^1\right) \tag{12.5}$$

With the help of figure 12.1, we can see that $\Delta_{(1)}u = \left(\Delta_{(1)}u^1, \Delta_{(1)}u^2\right)$, $\Delta_{(2)}u = \left(\Delta_{(2)}u^1, \Delta_{(2)}u^2\right)$ are tangents of the straight lines $a(t) = u + t\Delta_{(1)}u$, $a(t) = u + t\Delta_{(2)}u$, $t \in [0,1]$ at their common point *u*. These curves lie in the parameters' domain *B* and pass from the point *u*, for *t*=0. Hence, we infer that (see paragraph 10):

A) The vectors $\Delta_{(1)}U = e_\mu(u)\Delta_{(1)}u^\mu$, $\Delta_{(2)}U = e_\mu(u)\Delta_{(2)}u^\mu$ belong to the tangent space $T_u S$ of *S*.

B) Relations 12.3 and 12.5 are written in the language of the forms:

$$\xi_{(1)}{}^\mu - \xi_{(0)}{}^\mu = R^\mu_{\nu\lambda\kappa}\xi_{(0)}{}^\nu\omega^\lambda \wedge \omega^\kappa\left(\Delta_{(1)}U, \Delta_{(2)}U\right) \tag{12.6a}$$

$$\xi_{(1)}{}^\mu - \xi_{(0)}{}^\mu = \left(R^\mu_{\nu 12} - R^\mu_{\nu 21}\right)\xi_{(0)}{}^\nu\omega^1 \wedge \omega^2\left(\Delta_{(1)}U, \Delta_{(2)}U\right) \tag{12.6b}$$

C) The area of the infinitesimal region $\Delta\Pi_u \equiv \Delta\Pi_u[\Delta_{(1)}U, \Delta_{(2)}U]$ on *S* is given by the equation:

$$\text{area}\left(\Delta\Pi_u\right) = \sqrt{\det g(u)}\left(\omega^1 \wedge \omega^2\right)\left(\Delta_{(1)}U, \Delta_{(2)}U\right) = \sqrt{\det g(u)}\left(\Delta_{(1)}u^1\Delta_{(2)}u^2 - \Delta_{(1)}u^2\Delta_{(2)}u^1\right) \tag{12.6c}$$

From 12.6a-c we obtain the following expressions:

$$\xi_{(1)}{}^\mu - \xi_{(0)}{}^\mu = \frac{1}{\sqrt{\det g(u)}}\xi_{(0)}{}^\nu R^\mu_\nu\,\text{area}\left(\Delta\Pi_u\right) \tag{12.7a}$$

$R^\mu_\nu \underset{def}{=} R^\mu_{\nu 12} - R^\mu_{\nu 21}$

$$\xi_{(1)}{}^\mu - \xi_{(0)}{}^\mu = \frac{1}{\sqrt{\det g(u)}}\xi_{(0)}{}^\nu g^{\mu\kappa}R_{\kappa\nu}\,\text{area}\left(\Delta\Pi_u\right) \tag{12.7b}$$

$R_{\kappa\nu} = g_{\kappa\lambda}R^\lambda_\nu$

We call $\left[R_{\kappa\nu}(u)\right]$ "**curvature matrix**".

Properties of the curvature matrix

A) From the definition 12.4 and the relationship 12.7a we obtain the identity:

$$R^{\lambda}_{\nu} = R^{\lambda}_{\nu 12} - R^{\lambda}_{\nu 21} = \bar{\Gamma}^{\lambda}_{\rho 2}\bar{\Gamma}^{\rho}_{\nu 1} - \bar{\Gamma}^{\lambda}_{\rho 1}\bar{\Gamma}^{\rho}_{\nu 2} - \partial_1\bar{\Gamma}^{\lambda}_{\nu 2} + \partial_2\bar{\Gamma}^{\lambda}_{\nu 1} \tag{12.8a}$$

Assuming that the connection of the geometric surface *S* is symmetric and compatible with the metric tensor, we derive the general form of the curvature matrix:

By taking into account the identities 11.15, we obtain the subsequent relationships:

$$R_{\kappa\nu} = g_{\kappa\lambda}R^{\lambda}_{\nu} = g_{\kappa\lambda}\left(\bar{\Gamma}^{\lambda}_{\rho 2}\bar{\Gamma}^{\rho}_{\nu 1} - \bar{\Gamma}^{\lambda}_{\rho 1}\bar{\Gamma}^{\rho}_{\nu 2} - \partial_1\bar{\Gamma}^{\lambda}_{\nu 2} + \partial_2\bar{\Gamma}^{\lambda}_{\nu 1}\right) = -\partial_1\bar{\Gamma}_{\kappa\nu 2} + \partial_2\bar{\Gamma}_{\kappa\nu 1} - \bar{\Gamma}^{\lambda}_{\nu 1}\bar{\Gamma}_{\lambda\kappa 2} + \bar{\Gamma}^{\lambda}_{\nu 2}\bar{\Gamma}_{\lambda\kappa 1} \tag{12.8b}$$

$$R_{\nu\kappa} = -\partial_1\bar{\Gamma}_{\nu\kappa 2} + \partial_2\bar{\Gamma}_{\nu\kappa 1} - \bar{\Gamma}^{\lambda}_{\kappa 1}\bar{\Gamma}_{\lambda\nu 2} + \bar{\Gamma}^{\lambda}_{\kappa 2}\bar{\Gamma}_{\lambda\nu 1} \tag{12.8c}$$

$$R_{\kappa\nu} + R_{\nu\kappa} = 0 \tag{12.8d}$$

$$R_{11} = R_{22} = 0,\ R_{12} + R_{21} = 0 \tag{12.8e}$$

From 12.8e we imply that the curvature matrix has the form:

$$\left[R_{\kappa\nu}(u)\right] = R(u)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{12.9}$$

The real function $R(u)$ is determined by the expression:

$$R(u) = -\partial_1\bar{\Gamma}_{212} + \partial_2\bar{\Gamma}_{211} + \bar{\Gamma}_{\lambda 12}\bar{\Gamma}^{\lambda}_{12} - \bar{\Gamma}_{\lambda 22}\bar{\Gamma}^{\lambda}_{11} \tag{12.10}$$

B) **Transformation of the curvature matrix under a parameter-transformation**

Let $u^{\mu} = u^{\mu}(\tilde{u})$ be a parameter-transformation. *How does relation 12.7b transform?*

We have seen (see paragraph 10) that under $u^{\mu} = u^{\mu}(\tilde{u})$ the area-element of the geometric surface *S* is an invariant 2-form:

$\text{area}(\Delta\Pi_u) = \text{area}(\Delta\Pi_{\tilde{u}})$

We transform the other terms of 12.7b as we have already learnt and we obtain the expressions:

$$\frac{\partial u^{\mu}}{\partial\tilde{u}^{\alpha}}\left(\tilde{\xi}_{(1)}{}^{\alpha} - \tilde{\xi}_{(0)}{}^{\alpha}\right) = \frac{1}{\det\tilde{J}\sqrt{\det\tilde{g}}}\frac{\partial u^{\nu}}{\partial\tilde{u}^{\beta}}\tilde{\xi}_{(0)}{}^{\beta}\frac{\partial u^{\mu}}{\partial\tilde{u}^{\lambda}}\frac{\partial u^{\kappa}}{\partial\tilde{u}^{\rho}}\tilde{g}^{\lambda\rho}R_{\kappa\nu}\,\text{area}(\Delta\Pi_{\tilde{u}})$$

$$\det\tilde{J} = \frac{1}{\det J} = \det\begin{pmatrix} \frac{\partial\tilde{u}^1}{\partial u^1} & \frac{\partial\tilde{u}^1}{\partial u^2} \\ \frac{\partial\tilde{u}^2}{\partial u^1} & \frac{\partial\tilde{u}^2}{\partial u^2} \end{pmatrix} = \sqrt{\frac{\det g}{\det\tilde{g}}}$$

$$\tilde{\xi}_{(1)}{}^{\alpha} - \tilde{\xi}_{(0)}{}^{\alpha} = \frac{1}{\det\tilde{J}\sqrt{\det\tilde{g}}}\frac{\partial u^{\nu}}{\partial\tilde{u}^{\beta}}\tilde{\xi}_{(0)}{}^{\beta}\frac{\partial u^{\kappa}}{\partial\tilde{u}^{\rho}}\tilde{g}^{\alpha\rho}R_{\kappa\nu}\,\text{area}(\Delta\Pi_{\tilde{u}}) \underset{def}{=} \frac{1}{\sqrt{\det\tilde{g}}}\tilde{\xi}_{(0)}{}^{\beta}\tilde{g}^{\alpha\rho}\tilde{R}_{\rho\beta}\,\text{area}(\Delta\Pi_{\tilde{u}})$$

The matrix-elements of the curvature matrix transform according to the relationship:

$$\tilde{R}_{\rho\beta} = \frac{1}{\det\tilde{J}}\frac{\partial u^{\kappa}}{\partial\tilde{u}^{\rho}}\frac{\partial u^{\nu}}{\partial\tilde{u}^{\beta}}R_{\kappa\nu} \tag{12.13}$$

By combining 12.9 with 12.13 we obtain the following expression for the transformed curvature matrix:

$$\left[\tilde{R}_{\rho\beta}\right] = R(u)\frac{\det\tilde{g}(\tilde{u})}{\det g(u)}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \underset{def}{=} \tilde{R}(\tilde{u})\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{12.14a}$$

From 12.14a we imply that:

$$\frac{R(u)}{\det g(u)} = \frac{\tilde{R}(\tilde{u})}{\det\tilde{g}(\tilde{u})} \underset{def}{=} K \tag{12.14b}$$

The quantity K is independent of the choice of the system of the parameters; it depends only on the point $P \leftrightarrow \left(u^1, u^2\right) \leftrightarrow \left(\tilde{u}^1, \tilde{u}^2\right)$ of the geometric surface, we have considered. The real function $K(P)$ is a **geometric invariant** of the geometric surface S; it is called "**the curvature of the geometric surface at its point *P*"**.

According to 12.10, the curvature of a geometric surface at any point $P \in S$ is calculated by the relation:

$$K(P) = \frac{R}{\det g} = \frac{1}{\det g}\left(-\partial_1 \bar{\Gamma}_{212} + \partial_2 \bar{\Gamma}_{211} + \bar{\Gamma}_{\lambda 12}\bar{\Gamma}^{\lambda}_{12} - \bar{\Gamma}_{\lambda 22}\bar{\Gamma}^{\lambda}_{11}\right) \tag{12.15}$$

Example 12.A

Calculation of the curvature of a sphere

In this example we consider a geometric surface determined by a metric tensor arisen by the description of a sphere embedded in a 3-dimensional Euclidean space. The connection on this abstract sphere is defined to be symmetric and compatible with the metric tensor; we calculate the Christoffel symbols corresponding to this connection and then we derive the curvature at each point of the sphere by applying the general relation 12.15. It is confirmed the anticipated result, well-known by the elementary geometry of the 3-dimensional Euclidean space.

The surface of a sphere with radius b, embedded in the Euclidean space $\boldsymbol{R}^3_0$ could be considered as a geometric surface S determined by the following metric tensor:

$$g = \left[g_{\mu\nu}\right] = \begin{pmatrix} b^2 - (u^2)^2 & 0 \\ 0 & \dfrac{b^2}{b^2 - (u^2)^2} \end{pmatrix} \tag{12A.1a}$$

$u^1 \in \boldsymbol{R}$, $u^2 \in (-b, b)$

Remember that a sphere is to be considered as a surface of revolution. According to the general analytic expression of a surface of revolution, referred in Example 9A, for the case of a sphere we have:

$f(u^2) = \sqrt{b^2 - (u^2)^2}$

In this description, two diametrically opposite points of the sphere have been excluded. The analytic expression of a surface of revolution is periodic with respect to the parameter u^1. This periodicity for the geometric surface corresponding to a sphere is accomplished by imposing the following conditions on the basis-elements of its tangent spaces:

$$e_\mu\left(u^1, u^2\right) = e_\mu\left(u^1 + 2\pi, u^2\right),\ \mu = 1,2 \tag{12A.1b}$$

The basis $\left\{e_1(u), e_2(u)\right\}$ is compatible to the metric tensor 12A.1a, i.e.:

$$\left\langle e_\mu(u), e_\nu(u)\right\rangle = g_{\mu\nu}(u) \tag{12A.1c}$$

The determinant of g is:

$\det g = b^2$

The inverse of the metric tensor is:

$$g^{-1} = \left[g_{\mu\nu}\right]^{-1} = \frac{1}{b^2}\begin{pmatrix} \dfrac{b^2}{b^2 - (u^2)^2} & 0 \\ 0 & b^2 - (u^2)^2 \end{pmatrix}$$

The Christoffel symbols corresponding to the connection on S which is symmetric and compatible with the metric tensor 12A.1, are calculated by applying relations 11.16:

$$\bar{\Gamma}_{111} = \bar{\Gamma}_{122} = \bar{\Gamma}_{212} = \bar{\Gamma}_{221} = 0\,,\ \bar{\Gamma}_{112} = \bar{\Gamma}_{121} = -u^2\,,\ \bar{\Gamma}_{211} = u^2\,,\ \bar{\Gamma}_{222} = \frac{b^2u^2}{\left(b^2-(u^2)^2\right)} \tag{12A.2}$$

$$\begin{aligned} &\bar{\Gamma}^1_{11} = \bar{\Gamma}^1_{22} = \bar{\Gamma}^2_{12} = \bar{\Gamma}^2_{21} = 0,\ \bar{\Gamma}^1_{12} = \bar{\Gamma}^1_{21} = -\frac{u^2}{b^2-(u^2)^2} \\ &\bar{\Gamma}^2_{11} = \frac{u^2}{b^2}\left(b^2-(u^2)^2\right),\ \bar{\Gamma}^2_{22} = \frac{u^2}{b^2-(u^2)^2} \end{aligned} \tag{12A.3}$$

By applying 12.15, we obtain:

$$K(P) = \frac{R(u)}{\det g} = \frac{1}{b^2} \tag{12A.4}$$

Relation 12A.4 holds for every point P of the sphere S; as expected, the curvature of S is constant everywhere.

13. **Geodesic curves on a geometric surface**

In this paragraph we define the concept of the geodesic curves on a geometric surface: the tangent vectors of a geodesic curves are vector fields transported parallel to their-selves along the curve. Then an extremely important property of the geodesic curves comes out: locally, i.e. in an appropriate neighborhood of curves passing by two certain points of the surface, the length of the geodesic curve joining them is extreme.

Let us consider a certain curve $c(t) = \left(c^1(t), c^2(t)\right)$, $t \in I$ (I an interval of $\boldsymbol{R}$) lying in the parameter-space of the geometric surface S. Every couple $c(t)$ determines a point $P_{c(t)}$ on S; hence a curve C on S is determined, corresponding to c.
The set of the tangent vectors of C at its point $P_{c(t)}$ is the subspace of $T_{c(t)}S$ spanned by the vectors:

$e_\mu\left(c(t)\right)\dot{c}^\mu(t)\lambda$, $\lambda \in \boldsymbol{R}$, $\dot{c}^\mu(t) \underset{def}{=} \dfrac{dc^\mu(\mathrm{t})}{dt}$

The elementary length Δs of C is given by the norm of the infinitesimal tangent vector:
$\Delta C(t) = e_\mu\left(c(t)\right)\dot{c}^\mu(t)\Delta t$, $\Delta t \to 0$

$$\Delta s = \left\langle \Delta C(t), \Delta C(t)\right\rangle^{1/2} = \left(g_{\mu\nu}\left(c(t)\right)\dot{c}^\mu(t)\dot{c}^\nu(t)\right)^{1/2}\Delta t \tag{13.1}$$

As t runs I, any tangent vector $\xi_c(t) = e_\mu\left(c(t)\right)\dot{c}^\mu(t)$ of C defines a vector field along C. The variation of $\xi_c(t)$ along C is calculated by its covariant differential (relation 11.9b):

$$D_{(c)}\xi_c(t) = e_\mu\left(c(t)\right)\left(\frac{d^2c^\mu(t)}{dt^2} + \bar{\Gamma}^\mu_{\nu\kappa}\dot{c}^\nu(t)\dot{c}^\kappa(t)\right)\Delta t \tag{13.2}$$

We say that the curve C is a **geodesic** of the geometric surface S, if only any tangent vector field $\xi_c(t)$ of C, is transferred parallel to itself along C.
By this definition we imply that C is a geodesic curve if and only if any tangent vector field $\xi_c(t)$ of C satisfies the condition:

$$D_{(c)}\xi_c(t) = 0 \tag{13.3a}$$

Hence, the curve $c(t) = \left(c^1(t), c^2(t)\right)$ of the parameter-space corresponding to a geodesic of the surface S, is a solution of the differential equations:

$$\frac{d^2c^\mu(t)}{dt^2} + \bar{\Gamma}^\mu_{\nu\kappa}\dot{c}^\nu(t)\dot{c}^\kappa(t) = 0 \qquad (13.3b)$$

Equation 13.3b is a second order differential equation; any solution of 13.3b is uniquely specified by two initial conditions. For example, there is just one geodesic of *S* passing by the point P_u and having a given tangent vector at this point:

$\left(c^1(0), c^2(0)\right) = \left(u^1, u^2\right) = u$

$\left(\dot{c}^1(0), \dot{c}^2(0)\right) = \left(v^1, v^2\right)$

Where v^1, v^2 are given quantities.

The curve of locally minimum length passing by two points of a geometric surface is a geodesic

Consider two points $P_{u_{(1)}}$, $P_{u_{(2)}}$ of the geometric surface *S*. Assume that there exists a unique curve *C* of *S* passing by these points, such that the length of the segment of *C* between $P_{u_{(1)}}$ and $P_{u_{(2)}}$ is a local minimum:

Let us symbolize:

$C : P_{c(t)} \in S,\ c : c(t) = \left(c^1(t), c^2(t)\right) \in B \subset \boldsymbol{R}^2$

$c(t_1) = u_{(1)}, c(t_2) = u_{(2)}$

If we consider any other curve $\bar{C}$ passing by the points $P_{u_{(1)}}$, $P_{u_{(2)}}$ which is "near" *C*, the length of the segment of $\bar{C}$ between $P_{u_{(1)}}$ and $P_{u_{(2)}}$ is larger than the length of the corresponding segment of *C*.

We describe the curve $\bar{C}$ as follows:

$\bar{C} : P_{\bar{c}(t)} \in S,\ \bar{c} : \bar{c}(t) = \left(\bar{c}^1(t), \bar{c}^2(t)\right)$

The conditions of "nearness" and "passing by the two mentioned points" are satisfied by writing:

$\bar{c}^\mu(t) = c^\mu(t) + \varepsilon\eta^\mu(t)$

Where ε is a real number in an appropriate neighborhood of zero and $\eta^\mu(t)$, $\mu = 1,2$ are arbitrary real functions satisfying the conditions:

$$\eta^\mu(t_1) = \eta^\mu(t_2) = 0 \qquad (13.4)$$

We choose the parameter *t* to be the length of the segment of the curve *C*, from $P_{u_{(1)}}$ to any point $P_{c(t)}$ between the points $P_{u_{(1)}}$, $P_{u_{(2)}}$ of *C*.

$$t = \int_{P_{u(1)}}^{P_{c(t)}} ds = \int_{P_{u(1)}}^{P_{c(t)}} \sqrt{g_{\mu\nu}(c)\, dc^\mu dc^\nu}$$

Hence: $dt = ds$, $t_1 = 0$, $t_2 = s_2 = \int_{P_{u(1)}}^{P_{u(2)}} ds$

We are going to show that the curve *C* is a geodesic of *S*, i.e. $c^\mu(s)$ are solutions of the differential equations 13.3b.

Steps to the proof

As soon as we have chosen the length s of the curve C as the common parameter of the curves $\bar{C}$ joining the two fixed points $P_{u_{(1)}}$, $P_{u_{(2)}}$ of S, the following identities are true:

$$ds^2 = g_{\mu\nu}(c)dc^\mu dc^\nu \tag{13.5a}$$

$$1 = g_{\mu\nu}(c)\dot{c}^\mu \dot{c}^\nu \tag{13.5b}$$

$\dot{c}^\nu \underset{def}{=} \frac{dc^\nu}{ds}$

Consider any curve $\bar{C}$ which is near C and passes by the fixed points $P_{u_{(1)}}$, $P_{u_{(2)}}$ of S:

$c(0) = u_{(1)}$, $c(s_2) = u_{(2)}$

$\bar{C} : P_{c(s)} \in S$, $\bar{c} : \bar{c}^\mu(s) = c^\mu(s) + \varepsilon\eta^\mu(s)$

$$\eta^\mu(0) = \eta^\mu(s_2) = 0 \tag{13.5c}$$

The length of the segment of the curve $\bar{C}$ between $P_{u_{(1)}}$, $P_{u_{(2)}}$ is:

$$\bar{s}_2 = \int_{P_{u_{(1)}}}^{P_{u_{(2)}}} d\bar{s} \tag{13.5d}$$

The elementary length $\Delta\bar{s}$ on $\bar{C}$ is calculated by the following relationships (terms up to the first order with respect to ε have been kept):

$\Delta\bar{s}^2 = g_{\mu\nu}(\bar{c}(s))\dot{\bar{c}}(s)\dot{\bar{c}}(s)\Delta s^2 = g_{\mu\nu}(c(s)+\varepsilon\eta(s))\left(\dot{c}^\mu(s)+\varepsilon\dot{\eta}^\mu(s)\right)\left(\dot{c}^\nu(s)+\varepsilon\dot{\eta}^\nu(s)\right)\Delta s^2 \approx$

$\approx \Delta s^2\left(1+\varepsilon\left(\eta^\kappa\partial_\kappa g_{\mu\nu}\dot{c}^\mu\dot{c}^\nu + 2g_{\mu\kappa}\dot{c}^\mu\dot{\eta}^\kappa\right)\right)$

$$\Delta\bar{s} \approx \Delta s\sqrt{1+\varepsilon\left(\eta^\kappa\partial_\kappa g_{\mu\nu}\dot{c}^\mu\dot{c}^\nu + 2g_{\mu\kappa}\dot{c}^\mu\dot{\eta}^\kappa\right)} \approx \Delta s + \Delta s\frac{\varepsilon}{2}\left(\eta^\kappa\partial_\kappa g_{\mu\nu}\dot{c}^\mu\dot{c}^\nu + 2g_{\mu\kappa}\dot{c}^\mu\dot{\eta}^\kappa\right) \tag{13.5e}$$

Given that the segment-length of C is extreme, we have:

$$\lim_{\varepsilon\to 0}\left(\frac{\bar{s}_2 - s_2}{\varepsilon}\right) = 0 \tag{13.5f}$$

Hence, by a combination of the relations 13.5d-f, we imply that:

$$\int_0^{s_2} ds\left(\eta^\kappa\partial_\kappa g_{\mu\nu}\dot{c}^\mu\dot{c}^\nu + 2g_{\mu\kappa}\dot{c}^\mu\dot{\eta}^\kappa\right) = 0 \tag{13.6}$$

From 13.6 we obtain the subsequent equations:

$$\int_0^{s_2} ds\left(\eta^\kappa\partial_\kappa g_{\mu\nu}\dot{c}^\mu\dot{c}^\nu + \frac{d}{ds}\left(2g_{\mu\kappa}\dot{c}^\mu\eta^\kappa\right) - \eta^\kappa\frac{d}{ds}\left(2g_{\mu\kappa}\dot{c}^\mu\right)\right) = 0$$

$$\int_0^{s_2} ds\eta^\kappa\left(\partial_\kappa g_{\mu\nu}\dot{c}^\mu\dot{c}^\nu - 2\partial_\nu g_{\mu\kappa}\dot{c}^\nu\dot{c}^\mu - g_{\mu\kappa}\ddot{c}^\mu\right) = 0$$

The last equation is true for any real function $\eta^\mu(s)$ hence we infer that [5]:

$$g_{\mu\kappa}\ddot{c}^\mu + \frac{1}{2}\left(-\partial_\kappa g_{\mu\nu} + \partial_\mu g_{\nu\kappa} + \partial_\nu g_{\kappa\mu}\right)\dot{c}^\mu\dot{c}^\nu = 0$$

From this and 11.16, we result the desired result:

$$g_{\mu\kappa}\ddot{c}^\mu + \bar{\Gamma}_{\kappa\mu\nu}\dot{c}^\mu\dot{c}^\nu = 0$$

$$\ddot{c}^\lambda + \bar{\Gamma}^\lambda_{\mu\nu}\dot{c}^\mu\dot{c}^\nu = 0$$

•

14. **Frame fields and connection forms on a geometric surface**

Now, we try to simplify the formulations, especially with respect to the calculation of the curvature on a geometric surface and the variation of the angle formed by a vector field moving along a closed curve and the basis-elements of the corresponding tangent spaces. This is achieved by the construction of two vector fields that are orthonormal in every tangent space of the geometric surface. These vector fields consist what is usually called "a frame field" on the tangent spaces of the geometric surface [(2)]. The vectors of this "frame field" play the role of orthonormal basis-vectors at every tangent space of the geometric surface; their construction is accomplished by applying the Cram-Schmidt procedure of orthonormalization [(5), (7)]. We calculate the Christoffel symbols related to the defined frame field. We introduce the "connection forms", by using the concept of the covariant differentiation expressed in the frame field and we use them to express and calculate the curvature matrix of the geometric surface.

Consider a geometric surface S determined by the metric tensor:

$g(u) = \left[g_{\mu\nu}(u) \right] = \left[\left\langle e_\mu(u), e_\nu(u) \right\rangle \right]$

The metric tensor is invertible (paragraph 7); hence:

$\det g \neq 0$

Let us also assume that for every u it holds that:

$g_{11}(u) \neq 0$

Then we are possible to define at every tangent space $T_u S$ of S a linear transformation L_u determined by the following relations [(5), (7)]:

$$L_u\left(e_1\right) = \tilde{e}_1 = e_1 \frac{1}{\sqrt{g_{11}}} \tag{14.1a}$$

$$L_u\left(e_2\right) = \tilde{e}_2 = \frac{1}{\sqrt{\det g}} \left(-e_1 \frac{g_{12}}{\sqrt{g_{11}}} + e_2 \sqrt{g_{11}} \right) \tag{14.1b}$$

Relations 14.1a and b define the vector fields $\tilde{e}_1(u)$, $\tilde{e}_2(u)$ on the tangent spaces of S. For each u the vectors $\tilde{e}_1(u)$, $\tilde{e}_2(u)$ have unit norm and they are mutually orthogonal:

$\left\langle \tilde{e}_\mu(u), \tilde{e}_\nu(u) \right\rangle = \delta_{\mu\nu}$

In brief, we say that the defined vector field is orthonormal. Such a field, defined on the tangent spaces of S is called a "**frame field**" on S. The frame field determines an orthonormal basis on every tangent space of the geometric surface.

According to 14.1a and b, the matrix of the linear transformation L_u with respect to the initial basis is:

$$L_u = \left[L_\mu^\nu \right] = \begin{pmatrix} L_1^1 & L_2^1 \\ L_1^2 & L_2^2 \end{pmatrix} = \frac{1}{\sqrt{g_{11} \det g}} \begin{pmatrix} \sqrt{\det g} & -g_{12}\sqrt{g_{11}} \\ 0 & g_{11} \end{pmatrix} \tag{14.1c}$$

The matrix of the inverse transformation is:

$$\tilde{L}_u = \left[\tilde{L}_\mu^\nu \right] = \begin{pmatrix} \sqrt{g_{11}} & g_{12} \\ 0 & \sqrt{\dfrac{\det g}{g_{11}}} \end{pmatrix} \tag{14.1d}$$

Remark: The determinant of the matrix $L_u = \left[L_\mu^\nu \right]$ is a function of the determinant of the metric tensor only:

$$\left(\det L_u \right)^2 = \frac{1}{\det g(u)} \tag{14.1e}$$

Calculation of the Christoffel symbols in the defined frame field

How do the Christoffel symbols transform, under the change of the basis-vectors caused by the linear transformation L_u?

Under the linear transformation L_u the basis-vector field $\{e_1(u),e_2(u)\}$ of the tangent spaces T_uS transforms to the frame field $\{\tilde{e}_1(u),\tilde{e}_2(u)\}$ according to the relations 14.1a and b:

$$\tilde{e}_\nu(u) = e_\mu(u)L^\mu_\nu(u) \tag{14.2}$$

The matrix $\left[L^\nu_\mu(u)\right]$ of the linear transformation L_u is given by 14.1c.

The covariant differentials of the basis-vector field $\{e_1(u),e_2(u)\}$ along the infinitesimal variation $\Delta u = \left(\Delta u^1, \Delta u^2\right)$ of the parameters $u=(u^1,u^2)$ are given by the equations (relation 11.19):

$D_{\Delta u} e_\mu = e_\nu \bar{\Gamma}^\nu_{\mu\kappa}\Delta u^\kappa$

Similarly[2], the covariant differentials of the frame field $\{\tilde{e}_1(u),\tilde{e}_2(u)\}$ are determined by the symbols $\tilde{\bar{\Gamma}}^\nu_{\mu\kappa}$ according to the equations:

$$D_{\Delta u}\tilde{e}_\mu = \tilde{e}_\nu \tilde{\bar{\Gamma}}^\nu_{\mu\kappa}\Delta u^\kappa \tag{14.3}$$

We can verify the subsequent identities:

$D_{\Delta u}\left(e_\lambda L^\lambda_\mu\right) = e_\rho L^\rho_\nu \tilde{\bar{\Gamma}}^\nu_{\mu\kappa}\Delta u^\kappa$

$D_{\Delta u}\left(e_\lambda\right)L^\lambda_\mu + e_\lambda D_{\Delta u}\left(L^\lambda_\mu\right) = e_\rho L^\rho_\nu \tilde{\bar{\Gamma}}^\nu_{\mu\kappa}\Delta u^\kappa$

$D_{\Delta u}\left(e_\lambda\right)L^\lambda_\mu + e_\lambda \partial_\kappa L^\lambda_\mu \Delta u^\kappa = e_\rho L^\rho_\nu \tilde{\bar{\Gamma}}^\nu_{\mu\kappa}\Delta u^\kappa$

$e_\rho \bar{\Gamma}^\rho_{\lambda\kappa} L^\lambda_\mu \Delta u^\kappa + e_\rho \partial_\kappa L^\rho_\mu \Delta u^\kappa = e_\rho L^\rho_\nu \tilde{\bar{\Gamma}}^\nu_{\mu\kappa}\Delta u^\kappa$

$$\tilde{\bar{\Gamma}}^\nu_{\mu\kappa} = \tilde{L}^\nu_\rho \partial_\kappa L^\rho_\mu + \tilde{L}^\nu_\rho \bar{\Gamma}^\rho_{\lambda\kappa} L^\lambda_\mu \tag{14.4}$$

Relation 14.4 affords us to calculate the Christoffel symbols in the frame field $\{\tilde{e}_1(u),\tilde{e}_2(u)\}$ if we know them with respect to the original basis-field $\{e_1(u),e_2(u)\}$ which is compatible to the metric tensor $g(u)$ determining the geometric surface in the u-parameter system. Notice that no change of the parameters has been imposed; compare 14.4 with 11.12.

Remarks:

A) Expression 14.4 holds for any change of the basis-elements in the tangent spaces of S, whence keeping the parameters unaltered. Nevertheless, under the linear transformation 14.1 we have constructed two vector fields defined on S, which are orthonormal for every tangent space of S. That means that our surface is always a Euclidean plane? Of course not! This would be true if only for any two tangent spaces $T_{u_{(1)}}S$, $T_{u_{(2)}}S$ and for any curve a of the parameters' domain, joining $u_{(1)}$, $u_{(2)}$ the corresponding orthonormal basis-vectors are the parallel transport of each other, with respect to the symmetric connection $\bar{\varphi}$ i.e.:

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[2] The symbols $\tilde{\bar{\Gamma}}_{\mu\nu\kappa}$ and $\tilde{\bar{\Gamma}}^\lambda_{\nu\kappa}$ are determined by expressing the matrix-elements of the defined connection in the defined frame field. For $\Delta u \to (0,0)$ we have:

$$D_{\Delta u}\tilde{e}_\mu(u) = \bar{\varphi}_{u,u+\Delta u}\left(\tilde{e}_\mu(u+\Delta u)\right) - \tilde{e}_\mu(u) = \tilde{e}_\lambda(u)\left(\tilde{\bar{\varphi}}^\lambda_\mu(u,u+\Delta u) - \delta^\lambda_\mu\right) = \tilde{e}_\lambda(u)\left.\frac{\partial\tilde{\bar{\varphi}}^\lambda_\mu(u,v)}{\partial v^\kappa}\right|_{v=u}\Delta u^\kappa = \tilde{e}_\lambda(u)\tilde{\bar{\Gamma}}^\lambda_{\mu\kappa}\Delta u^\kappa$$

$\tilde{\bar{\varphi}}^\lambda_\mu$ are the matrix-elements of the connection with respect to the frame field: $\{\tilde{e}_1,\tilde{e}_2\}$

$\tilde{e}_\mu\left(u_{(2)}\right) = \bar{\varphi}^{(a)}_{u_{(2)},u_{(1)}}\left(\tilde{e}_\mu\left(u_{(1)}\right)\right)$

Or equivalently, for any infinitesimal variation $\Delta u = \left(\Delta u^1, \Delta u^2\right)$ of the u-parameters:

$D_{\Delta u}\tilde{e}_\mu = 0$

Of course, this is not true in general, as we can see by the relationships 14.3 and 4.

B) The fact that the vectors if the defined frame field are everywhere orthonormal, causes some interesting consequences on the form of the Christoffel symbols when calculated with respect to it. We depart from 14.3 and we obtain the subsequent identities:

$\left\langle \tilde{e}_\nu, \tilde{e}_\mu \right\rangle = \delta_{\nu\mu}$

$D_{\Delta u}\left(\left\langle \tilde{e}_\nu, \tilde{e}_\mu \right\rangle\right) = 0$

$\left\langle D_{\Delta u}\tilde{e}_\nu, \tilde{e}_\mu \right\rangle + \left\langle \tilde{e}_\nu, D_{\Delta u}\tilde{e}_\mu \right\rangle = 0$

$\left\langle \tilde{e}_\lambda, \tilde{e}_\mu \right\rangle \tilde{\bar{\Gamma}}^\lambda_{\nu\kappa}\Delta u^\kappa + \left\langle \tilde{e}_\nu, \tilde{e}_\lambda \right\rangle \tilde{\bar{\Gamma}}^\lambda_{\mu\kappa}\Delta u^\kappa = 0$

$\left(\delta_{\lambda\mu}\tilde{\bar{\Gamma}}^\lambda_{\nu\kappa} + \delta_{\nu\lambda}\tilde{\bar{\Gamma}}^\lambda_{\mu\kappa}\right)\Delta u^\kappa = 0$

(The symbol $\delta_{\lambda\mu}$ stands for the Kronecker delta)

The previous identity holds for any infinitesimal vector $\Delta u = \left(\Delta u^1, \Delta u^2\right)$ of the parameter-space; hence:

$$\tilde{\bar{\Gamma}}_{\mu\nu\kappa} + \tilde{\bar{\Gamma}}_{\nu\mu\kappa} = 0 \qquad (14.5a)$$

$\tilde{\bar{\Gamma}}_{\mu\nu\kappa} \underset{def}{=} \left\langle \tilde{e}_\lambda, \tilde{e}_\mu \right\rangle \tilde{\bar{\Gamma}}^\lambda_{\nu\kappa} = \delta_{\lambda\mu}\tilde{\bar{\Gamma}}^\lambda_{\nu\kappa}$

From 14.5a we result that the values of the symbols $\tilde{\bar{\Gamma}}_{\mu\nu\kappa}$, $\tilde{\bar{\Gamma}}^\nu_{\mu\kappa}$ are given by the following equations:

$$\tilde{\bar{\Gamma}}_{111} = \tilde{\bar{\Gamma}}_{112} = 0 \quad \tilde{\bar{\Gamma}}_{221} = \tilde{\bar{\Gamma}}_{222} = 0 \quad \tilde{\bar{\Gamma}}_{12\kappa} + \tilde{\bar{\Gamma}}_{21\kappa} = 0 \qquad (14.5b)$$

$$\tilde{\bar{\Gamma}}^1_{11} = \tilde{\bar{\Gamma}}^1_{12} = \tilde{\bar{\Gamma}}^2_{21} = \tilde{\bar{\Gamma}}^2_{22} = 0 \quad \tilde{\bar{\Gamma}}^1_{2\kappa} + \tilde{\bar{\Gamma}}^2_{1\kappa} = 0 \qquad (14.5c)$$

C) Notice that the symbols $\tilde{\bar{\Gamma}}_{\mu\nu\kappa}$ are no longer symmetric with respect to the last two indices ν,κ as is the case for the symbols $\bar{\Gamma}_{\mu\nu\kappa}$ ($\bar{\Gamma}_{\mu\nu\kappa} = \bar{\Gamma}_{\mu\kappa\nu}$). Notice that $\bar{\Gamma}_{\mu\nu\kappa}$ arise from the matrix-elements of the connection $\bar{\varphi}$ which is symmetric and compatible with the metric tensor $g(u)$; these matrix-elements have been calculated with respect to the basis-field $\{e_1(u), e_2(u)\}$ which obey the condition:

$g(u) = \left[g_{\mu\nu}(u)\right] = \left[\left\langle e_\mu(u), e_\nu(u)\right\rangle\right]$

Instead, by 14.5a we conclude that $\tilde{\bar{\Gamma}}_{\mu\nu\kappa}$ are antisymmetric with respect to the indices μ,ν -the initial two indices.

D) According to the footnote of the present paragraph, the symbols $\tilde{\bar{\Gamma}}^\lambda_{\mu\kappa}$ are defined by the relations:

$$\tilde{\bar{\Gamma}}^\lambda_{\mu\kappa} = \left.\frac{\partial \tilde{\bar{\varphi}}^\lambda_\mu(u,v)}{\partial v^\kappa}\right|_{v=u}$$

The matrix $\left[\tilde{\bar{\varphi}}^\lambda_\mu(u,v)\right]$, $v = u + \Delta u$, $\Delta u \to (0,0)$ is the matrix of the linear transform $\bar{\varphi}_{u,u+\Delta u} : T_{u+\Delta u}S \to T_u S$ with respect to the frame field $\left\{\tilde{e}_1(v), \tilde{e}_2(v)\right\}$, $v \in B$ defined by 14.1a and b.

According to the section "covariant differentiation in a geometric surface" of the paragraph 11, it holds:

$$\left[\tilde{\tilde{\varphi}}_{\mu}^{\lambda}(v,u)\right]=\left[\tilde{\tilde{\varphi}}_{\mu}^{\lambda}(u,v)\right]^{-1}$$

$$\left[\tilde{\tilde{\varphi}}_{\mu}^{\lambda}(u,u)\right]=\left[\delta_{\mu}^{\lambda}\right]$$

Hence we have:

$$\tilde{\tilde{\varphi}}_{\lambda}^{\mu}(u,u+\Delta u)\tilde{\tilde{\varphi}}_{\nu}^{\lambda}(u+\Delta u,u)=\delta_{\nu}^{\mu}$$

$$\left(\delta_{\lambda}^{\mu}+\left.\frac{\partial\tilde{\tilde{\varphi}}_{\lambda}^{\mu}(u,v)}{\partial v^{\kappa}}\right|_{v=u}\Delta u^{\kappa}\right)\left(\delta_{\nu}^{\lambda}+\left.\frac{\partial\tilde{\tilde{\varphi}}_{\nu}^{\lambda}(v,u)}{\partial v^{\kappa}}\right|_{v=u}\Delta u^{\kappa}\right)=\delta_{\nu}^{\mu}$$

$$\tilde{\tilde{\Gamma}}_{\nu\kappa}^{\mu}+\left.\frac{\partial\tilde{\tilde{\varphi}}_{\nu}^{\mu}(v,u)}{\partial v^{\kappa}}\right|_{v=u}=0$$

$$\tilde{\Gamma}_{\nu\kappa}^{\mu}\underset{def}{=}\left.\frac{\partial\tilde{\tilde{\varphi}}_{\nu}^{\mu}(v,u)}{\partial v^{\kappa}}\right|_{v=u}=-\tilde{\tilde{\Gamma}}_{\nu\kappa}^{\mu}$$

E) We confirm 14.5b by a direct calculation (see relation 14.14):

First we obtain an expression of $\tilde{\tilde{\Gamma}}_{\mu\nu\kappa}$ analogous to 14.4; the subsequent identities are easily verified:

$$\left\langle\tilde{e}_{\nu},D_{\Delta u}\tilde{e}_{\mu}\right\rangle=\tilde{\tilde{\Gamma}}_{\nu\mu\kappa}\Delta u^{\kappa}$$

$$\tilde{e}_{\mu}=e_{\lambda}L_{\mu}^{\lambda}$$

$$D_{\Delta u}\tilde{e}_{\mu}=D_{\Delta u}\left(e_{\beta}L_{\mu}^{\beta}\right)=e_{\alpha}\left(\bar{\Gamma}_{\lambda\kappa}^{\alpha}L_{\mu}^{\lambda}+\partial_{\kappa}L_{\mu}^{\alpha}\right)\Delta u^{\kappa}$$

$$\left\langle\tilde{e}_{\nu},D_{\Delta u}\tilde{e}_{\mu}\right\rangle=\left\langle e_{\beta},e_{\alpha}\right\rangle L_{\nu}^{\beta}\left(\bar{\Gamma}_{\lambda\kappa}^{\alpha}L_{\mu}^{\lambda}+\partial_{\kappa}L_{\mu}^{\alpha}\right)\Delta u^{\kappa}$$

$$\tilde{\tilde{\Gamma}}_{\nu\mu\kappa}=\bar{\Gamma}_{\beta\alpha\kappa}L_{\nu}^{\beta}L_{\mu}^{\alpha}+g_{\beta\alpha}L_{\nu}^{\beta}\partial_{\kappa}L_{\mu}^{\alpha}\qquad(14.5d)$$

From the last identity we shall confirm, for example, that:

$$\tilde{\tilde{\Gamma}}_{11\kappa}=0$$

We have:

$$\tilde{\tilde{\Gamma}}_{11\kappa}=\bar{\Gamma}_{\beta\alpha\kappa}L_{1}^{\beta}L_{1}^{\alpha}+g_{\beta\alpha}L_{1}^{\beta}\partial_{\kappa}L_{1}^{\alpha}=\bar{\Gamma}_{\beta\alpha\kappa}L_{1}^{\beta}L_{1}^{\alpha}+\partial_{\kappa}\left(g_{\beta\alpha}L_{1}^{\beta}L_{1}^{\alpha}\right)-\partial_{\kappa}g_{\beta\alpha}L_{1}^{\beta}L_{1}^{\alpha}-g_{\beta\alpha}\partial_{\kappa}L_{1}^{\beta}L_{1}^{\alpha}$$

$$\partial_{\kappa}\left(g_{\beta\alpha}L_{1}^{\beta}L_{1}^{\alpha}\right)=\partial_{\kappa}\left(\left\langle e_{\beta},e_{\alpha}\right\rangle L_{1}^{\beta}L_{1}^{\alpha}\right)=\partial_{\kappa}\left(\left\langle\tilde{e}_{1},\tilde{e}_{1}\right\rangle\right)=\partial_{\kappa}(1)=0$$

$$g_{\beta\alpha}\partial_{\kappa}L_{1}^{\beta}L_{1}^{\alpha}=g_{\alpha\beta}L_{1}^{\beta}\partial_{\kappa}L_{1}^{\alpha}=g_{\beta\alpha}L_{1}^{\beta}\partial_{\kappa}L_{1}^{\alpha}=\tilde{\tilde{\Gamma}}_{11\kappa}-\bar{\Gamma}_{\beta\alpha\kappa}L_{1}^{\beta}L_{1}^{\alpha}$$

$$\partial_{\kappa}g_{\beta\alpha}=\bar{\Gamma}_{\beta\alpha\kappa}+\bar{\Gamma}_{\alpha\beta\kappa}$$

$$2\tilde{\tilde{\Gamma}}_{11\kappa}=\bar{\Gamma}_{\beta\alpha\kappa}L_{1}^{\beta}L_{1}^{\alpha}-\left(\bar{\Gamma}_{\beta\alpha\kappa}+\bar{\Gamma}_{\alpha\beta\kappa}\right)L_{1}^{\beta}L_{1}^{\alpha}+\bar{\Gamma}_{\beta\alpha\kappa}L_{1}^{\beta}L_{1}^{\alpha}=-\bar{\Gamma}_{\alpha\beta\kappa}L_{1}^{\beta}L_{1}^{\alpha}+\bar{\Gamma}_{\alpha\beta\kappa}L_{1}^{\alpha}L_{1}^{\beta}=0$$

Connection forms related to the defined frame field

Consider the frame field $\{\tilde{e}_1(u),\tilde{e}_2(u)\}$ defined on the geometric surface *S*, according to the relations 14.1a and b. On the tangent spaces of *S*, we define the 1-forms $\tilde{\omega}_{\mu\nu}$ related with the frame field $\{\tilde{e}_1(u),\tilde{e}_2(u)\}$ according to the relationship:

$$\tilde{\omega}_{\mu\nu}(\Delta U)\underset{def}{=}\left\langle\tilde{e}_{\mu}(u),D_{\Delta u}\tilde{e}_{\nu}(u)\right\rangle,\ \Delta U=e_{\mu}(u)\Delta u^{\mu}\in T_{u}S\qquad(14.6a)$$

We combine 14.3, 5 and 6a and we result the identities:

$$\tilde{\omega}_{\mu\nu}(\Delta U)=\left\langle\tilde{e}_{\mu},D_{\Delta u}\tilde{e}_{\nu}\right\rangle=\tilde{\tilde{\Gamma}}_{\mu\nu\kappa}\Delta u^{\kappa}\qquad(14.6b)$$

$$\tilde{\omega}_{11}(\Delta U)=0 \quad \tilde{\omega}_{22}(\Delta U)=0$$
$$\tilde{\omega}_{12}(\Delta U)=\tilde{\tilde{\Gamma}}_{12\kappa}\Delta u^{\kappa}=-\tilde{\tilde{\Gamma}}_{21\kappa}\Delta u^{\kappa}=-\tilde{\omega}_{21}(\Delta U) \tag{14.6c}$$

We define the matrix-form:

$$\Omega=\begin{pmatrix}\tilde{\omega}_{11} & \tilde{\omega}_{12}\\ \tilde{\omega}_{21} & \tilde{\omega}_{22}\end{pmatrix}=\begin{pmatrix}0 & \tilde{\omega}_{12}\\ -\tilde{\omega}_{12} & 0\end{pmatrix} \tag{14.6d}$$

The matrix Ω is antisymmetric.

By using the notation of the defined 1-forms $\tilde{\omega}_{\mu\nu}$ the covariant differentials of the frame field $\{\tilde{e}_1,\tilde{e}_2\}$ (relations 14.3) are given by the equations:

$$D_{\Delta u}\tilde{e}_{\mu}=\tilde{e}_{\nu}\delta^{\nu\lambda}\tilde{\tilde{\Gamma}}_{\lambda\mu\kappa}\Delta u^{\kappa}=\tilde{e}_{\nu}\delta^{\nu\lambda}\tilde{\omega}_{\lambda\mu}(\Delta U)=\tilde{e}_1\tilde{\omega}_{1\mu}(\Delta U)+\tilde{e}_2\tilde{\omega}_{2\mu}(\Delta U)$$

$$\begin{aligned}D_{\Delta u}\tilde{e}_1&=\tilde{e}_2\tilde{\omega}_{21}(\Delta U)=-\tilde{e}_2\tilde{\omega}_{12}(\Delta U)\\ D_{\Delta u}\tilde{e}_2&=\tilde{e}_1\tilde{\omega}_{12}(\Delta U)\end{aligned} \tag{14.7}$$

Calculation of the curvature tensor in the defined frame field

Let us reconstruct the curvature tensor with respect to the frame field $\{\tilde{e}_1,\tilde{e}_2\}$ defined in 14.1a and b, by following the reasoning path of paragraph 12.

Consider an infinitesimal orthogonal parallelogram $\Delta\Pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)$ of the parameters' u-space with vertices the points:

$u=(u^1,u^2)$, $u+\Delta_{(1)}u$, $u+\Delta_{(1)}u+\Delta_{(2)}u$, $u+\Delta_{(2)}u$

$\Delta_{(1)}u=(\Delta u^1,0)$, $\Delta_{(2)}u=(0,\Delta u^2)$

The quantities Δu^1, Δu^2 are infinitesimals; in our calculations we keep terms up to the second order with respect to them.

The infinitesimal orthogonal parallelogram $\Delta\Pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)$ is mapped to an infinitesimal closed region[3] $\Delta\Pi_u$ of the geometric surface, with vertices the points:

P_u, $P_{u+\Delta_{(1)}u}$, $P_{u+\Delta_{(1)}u+\Delta_{(2)}u}$, $P_{u+\Delta_{(2)}u}$

We choose an arbitrary vector $\xi_{(0)}=\tilde{e}_{\mu}\tilde{\xi}_{(0)}{}^{\mu}\in T_uS$ and transport it parallel to itself along the boundary of the infinitesimal parallelogram.

Let $\xi(v)=\tilde{e}_{\mu}(v)\tilde{\xi}(v)^{\mu}$ be the image of $\xi_{(0)}$ at any tangent space: T_vS, $v\in\partial\Delta\Pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)$

The covariant derivative of the parallel displaced vector field $\xi(v)$ is zero:

$D_{\Delta u}\xi(v)=0$

Hence:

$$D_{\Delta u}\left(\tilde{e}_{\mu}\tilde{\xi}^{\mu}\right)=D_{\Delta u}\tilde{e}_{\mu}\tilde{\xi}^{\mu}+\tilde{e}_{\mu}D_{\Delta u}\tilde{\xi}^{\mu}=\tilde{e}_{\mu}\left(\tilde{\tilde{\Gamma}}^{\mu}_{\nu\kappa}\tilde{\xi}^{\nu}+\partial_{\kappa}\tilde{\xi}^{\mu}\right)\Delta u^{\kappa}=0$$

$$\tilde{\tilde{\Gamma}}^{\mu}_{\nu\kappa}\tilde{\xi}^{\nu}+\partial_{\kappa}\tilde{\xi}^{\mu}=0 \tag{14.8}$$

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[3] Given that the correspondence $\boldsymbol{R}^2\supset B\ni u\to P_u\in S$ is one to one, onto and continuous, any compact set C of the domain is mapped to a compact set $P(C)$ of the range; the boundary of C is mapped to the boundary of $P(C)$ (4).

The variation of the parallel transported vector field $\xi(v)$ under an infinitesimal variation of the parameters along the boundary of the parallelogram $\Delta\pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)$ is given by the equation:

$$\tilde{\xi}(v+\Delta u)=\bar{\varphi}_{v+\Delta u,v}\left(\tilde{\xi}(v)\right) \tag{14.9}$$

From 14.9, we are getting the subsequent equations:

$\tilde{e}_\mu(v+\Delta u)\tilde{\xi}^\mu(v+\Delta u)=\bar{\varphi}_{v+\Delta u,v}\left(\tilde{e}_\lambda(v)\right)\tilde{\xi}^\lambda(v)=\tilde{e}_\mu(v+\Delta u)\tilde{\bar{\varphi}}^\mu_\lambda(v+\Delta u,v)\tilde{\xi}^\lambda(v)$

$\tilde{\xi}^\mu(v+\Delta u)=\tilde{\xi}^\mu(v)-\tilde{\bar{\Gamma}}^\mu_{\lambda\kappa}(v)\tilde{\xi}^\lambda(v)\Delta u^\kappa$

$D_{\Delta u}\tilde{\xi}^\mu(v)=\tilde{\xi}^\mu(v+\Delta u)-\tilde{\xi}^\mu(v)=-\tilde{\bar{\Gamma}}^\mu_{\lambda\kappa}(v)\tilde{\xi}^\lambda(v)\Delta u^\kappa$

We write: $\Delta U=e_\mu(v)\Delta u^\mu\in T_vS$

Then, from the previous identity we can define the 1-forms:

$D_{(\Delta U)}\tilde{\xi}^\mu(v)\underset{def}{=}-\tilde{\bar{\Gamma}}^\mu_{\lambda\kappa}(v)\tilde{\xi}^\lambda(v)\omega^\kappa(\Delta U)$, $\mu=1,2$

$D_{()}\tilde{\xi}^\mu(v)\underset{def}{=}-\tilde{\bar{\Gamma}}^\mu_{\lambda\kappa}(v)\tilde{\xi}^\lambda(v)\omega^\kappa$, $\mu=1,2$

The 1-forms ω^κ are the already defined basic 1-forms (paragraph 8):

$\omega^\kappa(\Delta U)=\omega^\kappa\left(e_\mu\Delta u^\mu\right)=\Delta u^\kappa$

We integrate the 1-form $D_{()}\tilde{\xi}^\mu(v)$ along the image of $\partial\pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)$ on S:

$$\tilde{\xi}_{(1)}{}^\mu(u)-\tilde{\xi}_{(0)}{}^\mu(u)=-\oint_{\partial\pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)}\tilde{\bar{\Gamma}}^\mu_{\lambda\kappa}(v)\tilde{\xi}^\lambda(v)\omega^\kappa \tag{14.9a}$$

We symbolize $\xi_{(1)}(u)=\tilde{e}_\mu(u)\tilde{\xi}_{(1)}{}^\mu(u)\in T_uS$ the image of $\xi_{(0)}(u)$ when the journey ends up and the parallel transporting field returns to the original tangent space T_uS from which it has been departed.

From 14.9a, by using Stokes-theorem (paragraph 8) and taking into account 14.8, we derive the subsequent identities:

$$\tilde{\xi}_{(1)}{}^\mu(u)-\tilde{\xi}_{(0)}{}^\mu(u)=-\oint_{\partial\pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)}\tilde{\bar{\Gamma}}^\mu_{\lambda\kappa}(v)\tilde{\xi}^\lambda(v)\omega^\kappa=-\int_{\pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)}d\left(\tilde{\bar{\Gamma}}^\mu_{\lambda\kappa}(v)\tilde{\xi}^\lambda(v)\omega^\kappa\right)$$

$$=-\int_{\pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)}d\left(\tilde{\bar{\Gamma}}^\mu_{\lambda\kappa}(v)\tilde{\xi}^\lambda(v)\right)\wedge\omega^\kappa=-\int_{\pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)}\partial_\rho\left(\tilde{\bar{\Gamma}}^\mu_{\lambda\kappa}(v)\tilde{\xi}^\lambda(v)\right)\omega^\rho\wedge\omega^\kappa\approx$$

$$\approx-\partial_\rho\left(\tilde{\bar{\Gamma}}^\mu_{\lambda\kappa}(v)\tilde{\xi}^\lambda(v)\right)\Big|_{v=u}\omega^\rho\wedge\omega^\kappa\left(\Delta_{(1)}U\wedge\Delta_{(2)}U\right)=$$

$$=-\partial_\rho\left(\tilde{\bar{\Gamma}}^\mu_{\lambda\kappa}(v)\tilde{\xi}^\lambda(v)\right)\Big|_{v=u}\Delta_{(1)}u^\rho\wedge\Delta_{(2)}u^\kappa=\left(\tilde{\bar{\Gamma}}^\mu_{\sigma\kappa}\tilde{\bar{\Gamma}}^\sigma_{\lambda\rho}-\partial_\rho\tilde{\bar{\Gamma}}^\mu_{\lambda\kappa}\right)\tilde{\xi}_{(0)}{}^\lambda\Delta_{(1)}u^\rho\wedge\Delta_{(2)}u^\kappa$$

Given that the parallelogram $\pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)$ is infinitesimal, we have kept orders up to the 2nd order with respect to the variations of the parameters; the 1st order terms have been identically vanished.

We define the quantities:

$$\tilde{R}^\mu_{\lambda\rho\kappa}=\tilde{\bar{\Gamma}}^\sigma_{\lambda\rho}\tilde{\bar{\Gamma}}^\mu_{\sigma\kappa}-\partial_\rho\tilde{\bar{\Gamma}}^\mu_{\lambda\kappa} \tag{14.9b}$$

We conclude that:

$$\tilde{\xi}_{(1)}{}^\mu(u)-\tilde{\xi}_{(0)}{}^\mu(u)=\left(\tilde{R}^\mu_{\lambda12}(u)-\tilde{R}^\mu_{\lambda21}(u)\right)\tilde{\xi}_{(0)}{}^\lambda(u)\Delta_{(1)}u^1\Delta_{(2)}u^2 \tag{14.9c}$$

Remember that we have assumed that:

$\Delta_{(1)}u=\left(\Delta u^1,0\right)$, $\Delta_{(2)}u=\left(0,\Delta u^2\right)$

The area of the curved image $\Delta\Pi_u$ of the infinitesimal orthogonal parallelogram $\Delta\pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)$ is given by the relationship:

$$\text{area}\left(\Delta\Pi_u\right)=\sqrt{\det g(u)}\omega^1\wedge\omega^2\left(\Delta_{(1)}U,\Delta_{(2)}U\right)=\sqrt{\det g(u)}\Delta_{(1)}u^1\Delta_{(2)}u^2$$

$$\Delta_{(1)}U=e_\mu(u)\Delta_{(1)}u^\mu\ ,\ \Delta_{(2)}U=e_\mu(u)\Delta_{(2)}u^\mu\ ,\ g(u)=\left[\left\langle e_\mu(u),e_\nu(u)\right\rangle\right]$$

Hence, 14.9c takes the form:

$$\tilde{\xi}_{(1)}{}^\mu(u)-\tilde{\xi}_{(0)}{}^\mu(u)=\frac{1}{\sqrt{\det g(u)}}\tilde{R}^\mu_\lambda(u)\tilde{\xi}_{(0)}{}^\lambda(\text{u})\,\text{area}\left(\Delta\Pi_u\right)\qquad(14.10a)$$

The matrix $\left[\tilde{R}^\mu_\lambda\right]$ is defined by the relationship:

$$\tilde{R}^\mu_\lambda=\tilde{R}^\mu_{\lambda12}-\tilde{R}^\mu_{\lambda21}$$

By making use of 14.5c and 14.9b, after some tedious calculations we come up to the following results:

$$\tilde{R}^\mu_\lambda=\tilde{R}^\mu_{\lambda12}-\tilde{R}^\mu_{\lambda21}=\tilde{\tilde{\Gamma}}^\sigma_{\lambda1}\tilde{\tilde{\Gamma}}^\mu_{\sigma2}-\tilde{\tilde{\Gamma}}^\sigma_{\lambda2}\tilde{\tilde{\Gamma}}^\mu_{\sigma1}-\partial_1\tilde{\tilde{\Gamma}}^\mu_{\lambda2}+\partial_2\tilde{\tilde{\Gamma}}^\mu_{\lambda1}$$

$$\tilde{R}^1_1=\tilde{\tilde{\Gamma}}^2_{11}\tilde{\tilde{\Gamma}}^1_{22}-\tilde{\tilde{\Gamma}}^2_{12}\tilde{\tilde{\Gamma}}^1_{21}=0$$

$$\tilde{R}^2_2=\tilde{\tilde{\Gamma}}^1_{21}\tilde{\tilde{\Gamma}}^2_{12}-\tilde{\tilde{\Gamma}}^1_{22}\tilde{\tilde{\Gamma}}^2_{11}=0$$

$$\tilde{R}^1_2=-\partial_1\tilde{\tilde{\Gamma}}^1_{22}+\partial_2\tilde{\tilde{\Gamma}}^1_{21}\underset{def}{=}-\tilde{R}(u)$$

$$\tilde{R}^2_1=-\partial_1\tilde{\tilde{\Gamma}}^2_{12}+\partial_2\tilde{\tilde{\Gamma}}^2_{11}=\partial_1\tilde{\tilde{\Gamma}}^1_{22}-\partial_2\tilde{\tilde{\Gamma}}^1_{21}=-\tilde{R}^1_2=\tilde{R}(u)$$

$$\left[\tilde{R}^\mu_\lambda\right]=\tilde{R}(u)\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

Relation 14.10 has been expressed in the orthonormal basis $\{\tilde{e}_1,\tilde{e}_2\}$ where the metric tensor $\left[\tilde{g}_{\mu\nu}\right]$ at any tangent space T_uS equals to the identity-matrix:

$$\tilde{g}_{\mu\nu}(u)=\left\langle\tilde{e}_\mu(\text{u}),\tilde{e}_\nu(u)\right\rangle=\delta_{\mu\nu}$$

In any case, the matrix $\left[\tilde{R}_{\kappa\nu}\right]$ is related with $\left[\tilde{R}^\mu_\lambda\right]$ by the identities:

$$\tilde{R}_{\mu\nu}=\tilde{g}_{\mu\lambda}\tilde{R}^\lambda_\nu=\left\langle\tilde{e}_\mu,\tilde{e}_\lambda\right\rangle\tilde{R}^\lambda_\nu$$

$$\tilde{R}^\mu_\nu=\tilde{g}^{\mu\kappa}\tilde{R}_{\kappa\nu}$$

$$\left[\tilde{g}^{\mu\kappa}\right]=\left[\tilde{g}_{\mu\nu}\right]^{-1}$$

In our case:

$$\left[\tilde{g}^{\mu\kappa}\right]=\left[\tilde{g}_{\mu\nu}\right]=\left[\delta_{\mu\nu}\right]$$

Hence we can easily verify that:

$$\left[\tilde{R}_{\mu\nu}\right]=\tilde{R}(u)\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

Where:

$$\tilde{R}(u)=-\partial_1\tilde{\tilde{\Gamma}}_{212}+\partial_2\tilde{\tilde{\Gamma}}_{211}=\partial_1\tilde{\tilde{\Gamma}}_{122}-\partial_2\tilde{\tilde{\Gamma}}_{121}\qquad(14.10b)$$

$$\tilde{\tilde{\Gamma}}^\mu_{\nu\kappa}=\tilde{g}^{\mu\lambda}\tilde{\tilde{\Gamma}}_{\lambda\nu\kappa}=\delta^{\mu\lambda}\tilde{\tilde{\Gamma}}_{\lambda\nu\kappa}=\tilde{\tilde{\Gamma}}_{\mu\nu\kappa}$$

Finally, 14.10a takes the form (compare with 12.7a,b):

$$\tilde{\xi}_{(1)}{}^\mu(u)-\tilde{\xi}_{(0)}{}^\mu(u)=\frac{1}{\sqrt{\det g}}\tilde{g}^{\mu\kappa}\tilde{R}_{\kappa\lambda}\tilde{\xi}_{(0)}{}^\lambda\,\text{area}\left(\Delta\Pi_u\right)\qquad(14.11)$$

Remark: We quote the following relationships which are compatible with the formalism used in the present paragraph. The verification of these relationships is not difficult to be accomplished; do it as an exercise.

Let $u^\mu = u^\mu(\bar{u})$ be a parameters' transformation:

$$e_\mu(u)\xi^\mu(u) = \bar{e}_\mu(\bar{u})\bar{\xi}^\mu(\bar{u}) = \tilde{e}_\mu(u)\tilde{\xi}^\mu(u)$$

$$\tilde{g}_{\mu\nu} = \langle \tilde{e}_\mu, \tilde{e}_\nu \rangle = \delta_{\mu\nu},\ g_{\mu\nu} = \langle e_\mu, e_\nu \rangle,\ \bar{g}_{\mu\nu} = \langle \bar{e}_\mu, \bar{e}_\nu \rangle$$

$$\tilde{e}_\nu(u) = e_\mu(u)L^\mu_\nu(u),\ \left[L^\mu_\nu(u)\right]^{-1} = \left[\tilde{L}^\mu_\nu(u)\right]$$

$$\tilde{\xi}^\mu(u) = \tilde{L}^\mu_\nu(u)\xi^\nu(u)$$

$$\xi^\mu = \frac{\partial u^\mu}{\partial \bar{u}^\kappa}\bar{\xi}^\kappa = J^\mu_\kappa \bar{\xi}^\kappa,\ J^\mu_\kappa \underset{def}{=} \frac{\partial u^\mu}{\partial \bar{u}^\kappa},\ \left[J^\mu_\kappa\right]^{-1} \underset{def}{=} \left[\bar{J}^\mu_\kappa\right]$$

$$e_\mu = \bar{e}_\kappa \frac{\partial \bar{u}^\kappa}{\partial u^\mu} = \bar{e}_\kappa \bar{J}^\kappa_\mu$$

$$\tilde{\xi}^\mu = \tilde{L}^\mu_\nu J^\nu_\kappa \bar{\xi}^\kappa = \left(\tilde{L}J\right)^\mu_\kappa \bar{\xi}^\kappa = Q^\mu_\kappa \bar{\xi}^\kappa,\ \tilde{Q} \underset{def}{=} \tilde{L}J,\ Q \underset{def}{=} \tilde{Q}^{-1} = \bar{J}L$$

$$\tilde{g}_{\mu\nu} = \langle \tilde{e}_\mu, \tilde{e}_\nu \rangle = \langle e_\alpha, e_\beta \rangle L^\alpha_\mu L^\beta_\nu = \langle \bar{e}_\kappa, \bar{e}_\lambda \rangle \bar{J}^\kappa_\alpha \bar{J}^\lambda_\beta L^\alpha_\mu L^\beta_\nu = \bar{g}_{\kappa\lambda} Q^\kappa_\mu Q^\lambda_\nu$$

$$\tilde{g}^{\mu\nu} = \bar{g}^{\kappa\lambda}\tilde{Q}^\mu_\kappa \tilde{Q}^\nu_\lambda$$

$$\det Q = \frac{\det J}{\det L}$$

$$\det \tilde{g} = \det \bar{g} \frac{(\det L)^2}{(\det J)^2} = \frac{\det \bar{g}}{(\det Q)^2} = 1$$

$$\det g = \det \bar{g} (\det J)^{-2} = (\det L)^{-2}$$

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How do 14.10 and 11 change under a parameters' transformation?

We follow the reasoning path of the paragraph 12: we are thinking deeper into the equations 14.10a and 14.11 hoping to find out a quantity that is invariant under any diffeomorphic transformation of the parameters.

We transform 14.10a so that the coordinates of the vectors in each side are evaluated in the initial basis-field $\{e_1(u), e_2(u)\}$.

$$\tilde{\xi}^\mu(u) = \tilde{L}^\mu_\nu(u)\xi^\nu(u)$$

$$\xi_{(1)}{}^\mu(u) - \xi_{(0)}{}^\mu(u) = \frac{1}{\sqrt{\det g}} L^\mu_\alpha \tilde{R}^\alpha_\lambda \tilde{L}^\lambda_\beta \tilde{\xi}_{(0)}{}^\beta \operatorname{area}(\Delta\Pi_u)$$

Comparing this relation with 12.7a, we imply that:

$$R^\mu_\nu = L^\mu_\alpha \tilde{R}^\alpha_\beta \tilde{L}^\beta_\nu$$

$$R_{\mu\nu} = g_{\mu\kappa} L^\kappa_\alpha \tilde{R}^\alpha_\beta \tilde{L}^\beta_\nu \qquad (14.12)$$

We have seen that

$$\left[\tilde{R}^\mu_\lambda\right] = \tilde{R}(u)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\left[R_{\mu\nu}\right] = R(u)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

According to 12.14b the quantity $R(u)$ is related to the curvature $K(P)$ at the point P of S with the relation:

$$R(u) = K(u)\det g(u) \qquad (14.13a)$$

In paragraph 12 we have seen that the curvature at any point P of S is invariant under any parameter-transformation:

$K(P) \underset{def}{=} K(u) = \bar{K}(\bar{u})$ $P \leftrightarrow u = \left(u^1, u^2\right) \leftrightarrow \bar{u} = \left(\bar{u}^1, \bar{u}^2\right)$, $P \equiv P_u \equiv P_{\bar{u}}$

We wright 14.12 in matrix-form and equate the determinants of each side:

$$R(u)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \tilde{R}(u)\, g\, L^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} L^{-1}$$

$$R^2 = \tilde{R}^2 \det g \det L \frac{1}{\det L}$$

We combine this with 14.13a and we result that the curvature at P is calculated by the analytic expression:

$$K(P) = \frac{\tilde{R}(u)}{\sqrt{\det g(u)}} \tag{14.13b}$$

The value of the function $\tilde{R}(u)$ is given by 14.10b:

$$\tilde{R}(u) = -\partial_1 \tilde{\tilde{\Gamma}}_{212} + \partial_2 \tilde{\tilde{\Gamma}}_{211} = \partial_1 \tilde{\tilde{\Gamma}}_{122} - \partial_2 \tilde{\tilde{\Gamma}}_{121}$$

In matrix-form 14.11 is written as follows:

$$\begin{pmatrix} \tilde{\xi}_{(1)}{}^1 \\ \tilde{\xi}_{(1)}{}^2 \end{pmatrix} = \begin{pmatrix} \tilde{\xi}_{(0)}{}^1 \\ \tilde{\xi}_{(0)}{}^2 \end{pmatrix} + K(u)\,\mathrm{area}\left(\Delta\Pi_u\right)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{\xi}_{(0)}{}^1 \\ \tilde{\xi}_{(0)}{}^2 \end{pmatrix} \tag{14.14}$$

Example 14A

Application for the case of a spherical surface

The geometric surface corresponding to a sphere S of radius b, in a system of parameters $u=(u^1,u^2)$ is determined by the metric tensor (see example 12A):

$$g = \left[g_{\mu\nu}\right] = \begin{pmatrix} b^2 - (u^2)^2 & 0 \\ 0 & \dfrac{b^2}{b^2 - (u^2)^2} \end{pmatrix} \tag{14A.1a}$$

The domain of the parameters u^1,u^2 is identified by the relations:

$u^1 \in \boldsymbol{R}$, $u^2 \in (-b,b)$

The sphere is a surface of revolution; hence the correspondence $u = \left(u^1, u^2\right) \to P_u \in S$ is periodic (period$=2\pi$) with respect to the parameter u^1. As a consequence, the tangent spaces $T_{(u^1,u^2)}S$, $T_{(u^1+2\pi,u^2)}S$ are identical:

$T_{(u^1,u^2)}S \equiv T_{(u^1+2\pi,u^2)}S$

The vectors of the basis-vector field $e_\mu\left(u^1,u^2\right)$ corresponding to the metric tensor $g(u)$, are periodic functions with respect to the variable u^1:

$$e_\mu\left(u^1,u^2\right) = e_\mu\left(u^1 + 2\pi, u^2\right) \tag{14A.1b}$$

From 14A.1 we imply that:

$$\left\langle e_1, e_2 \right\rangle = \left\langle e_2, e_1 \right\rangle = 0,\ \left\langle e_1, e_1 \right\rangle = b^2 - (u^2)^2,\ \left\langle e_2, e_2 \right\rangle = \frac{b^2}{b^2 - (u^2)^2} \tag{14A.2a}$$

On S we have defined the symmetric connection $\bar{\varphi}$ which is compatible to the metric tensor 14A.1a. The corresponding Christoffel symbols with respect to the basis-vector field $\{e_1(u),e_2(u)\}$, have been calculated in the Example 12A, relations 12A.2, 3.

In the present Example, we use the already defined frame field $\{\tilde{e}_1(u), \tilde{e}_2(u)\}$ (see relations 14.1a and b) and we express the matrix-elements of $\bar{\varphi}$ with respect to this basis field. We calculate the curvature of the sphere by applying relation 14.13 and compared the result with 12A.4.

The elements of the frame field are determined by 14.2:

$\tilde{e}_\mu = e_\lambda L^\lambda_\mu$

$$L = \left[L^\nu_\mu\right] = \begin{pmatrix} L^1_1 & L^1_2 \\ L^2_1 & L^2_2 \end{pmatrix} = \frac{1}{\sqrt{g_{11}\det g}} \begin{pmatrix} \sqrt{\det g} & -g_{12}\sqrt{g_{11}} \\ 0 & g_{11} \end{pmatrix} \qquad (14A.2b)$$

According to 14A.1a:

$$L = \frac{1}{b\sqrt{b^2-(u^2)^2}} \begin{pmatrix} b & 0 \\ 0 & b^2-(u^2)^2 \end{pmatrix}$$
$$\tilde{e}_1 = e_1 \frac{1}{\sqrt{b^2-(u^2)^2}},\ \tilde{e}_2 = e_2 \frac{\sqrt{b^2-(u^2)^2}}{b} \qquad (14A.2c)$$

The inverse matrix of L is given by the expression:

$$\tilde{L} \underset{def}{=} L^{-1} = \frac{1}{\sqrt{b^2-(u^2)^2}} \begin{pmatrix} b^2-(u^2)^2 & 0 \\ 0 & b \end{pmatrix} \qquad (14A.2d)$$

The determinants of L and $\tilde{L}$ are:

$$\det L = 1/b,\ \det \tilde{L} = 1/\det L = b \qquad (14A.2e)$$

We notice that L and $\tilde{L}$ are independent on u^1. Hence the frame field $\{\tilde{e}_1(u), \tilde{e}_2(u)\}$ maintains the periodicity of the original basis-vector field, with respect to the parameter u^1:

$$\tilde{e}_\mu\left(u^1, u^2\right) = \tilde{e}_\mu\left(u^1 + 2\pi, u^2\right) \qquad (14A.3)$$

We calculate the Christoffel symbols with respect to $\{\tilde{e}_1, \tilde{e}_2\}$ by using 14.5b and 14.5d:

$\tilde{\bar{\Gamma}}_{\nu\mu\kappa} = \bar{\Gamma}_{\beta\alpha\kappa} L^\beta_\nu L^\alpha_\mu + g_{\beta\alpha} L^\beta_\nu \partial_\kappa L^\alpha_\mu$

We find that:

$$\tilde{\bar{\Gamma}}_{111} = \tilde{\bar{\Gamma}}_{112} = 0 \quad \tilde{\bar{\Gamma}}_{221} = \tilde{\bar{\Gamma}}_{222} = 0$$
$$\tilde{\bar{\Gamma}}_{12\kappa} + \tilde{\bar{\Gamma}}_{21\kappa} = 0 \qquad (14A.4)$$
$$\tilde{\bar{\Gamma}}_{121} = -\tilde{\bar{\Gamma}}_{211} = -\frac{u^2}{b} \quad \tilde{\bar{\Gamma}}_{122} = -\tilde{\bar{\Gamma}}_{212} = 0$$

Remark: The basis vectors of the frame field have everywhere unit length and they are orthogonal. However they do not transport parallel to themselves: from (14.3) we can easily verify that:

$D_{\Delta u}\tilde{e}_1 = \tilde{e}_2 \frac{u^2}{b} \Delta u^1$

$D_{\Delta u}\tilde{e}_2 = -\tilde{e}_1 \frac{u^2}{b} \Delta u^1$

We are now ready to calculate the curvature at every point of the surface, by applying 14.13; we only need the value of the quantity $\tilde{R}(u)$ given by 14.10b:

$\tilde{R}(u) = -\partial_1 \tilde{\bar{\Gamma}}_{212} + \partial_2 \tilde{\bar{\Gamma}}_{211} = \partial_1 \tilde{\bar{\Gamma}}_{122} - \partial_2 \tilde{\bar{\Gamma}}_{121}$

$$\tilde{R}(u) = -\partial_1 \tilde{\tilde{\Gamma}}_{212} + \partial_2 \tilde{\tilde{\Gamma}}_{211} = \frac{\partial}{\partial u^2}\left(\frac{u^2}{b}\right) = \frac{1}{b}$$

Hence:

$$K(P) = \frac{\tilde{R}(u)}{\sqrt{\det g(u)}} = \frac{1/b}{\sqrt{b^2}} = \frac{1}{b^2}$$

This is the expected result (see Example 12A).

Example 14B

Variation of the angle formed by a parallel transported vector field along a closed curve, with the corresponding vectors of a specific frame field

Application for the case of a sphere

In the section A of the present example, we verify that the norm of any parallel displaced vector field along the boundary of an infinitesimal parallelogram of the parameters' space is invariant. This result is accomplished by expressing the coordinates of the vector field with respect of the frame field defined by 14.1a and b and applying relation 14.14.
In section B we calculate the variation of the angle formed by the parallel displaced vector field with one of the corresponding elements of the frame field along the mentioned infinitesimal loop. We come to the very interesting result that the total variation of this angle is proportional to the curvature of the geometric surface at the vertex of the infinitesimal loop.
Finally, in the section C of the example, we apply the general equations derived in section B, on a geometric surface corresponding to a sphere. The objective is to verify the general relations for that special case, by following different reasoning paths and comparing the results.

A. According to 14.14, the variation of any vector $\xi_{(0)}(u) \in T_u S$ transported parallel to itself along the boundary $\partial \Pi_u\left(\Delta_{(1)}u, \Delta_{(2)}u\right)$ of the infinitesimal parallelogram $\Pi_u\left(\Delta_{(1)}u, \Delta_{(2)}u\right)$ of the parameters' space, is given by the equation:

$$\xi_{(1)} = \xi_{(0)} + \left(-\tilde{e}_1 \tilde{\xi}_{(0)}{}^2 + \tilde{e}_2 \tilde{\xi}_{(0)}{}^1\right) K \,\text{area}\left(\Delta\Pi_u\right) \qquad (14B.1)$$

$\xi_{(0)}(u) = \tilde{e}_\mu(u)\tilde{\xi}_{(0)}{}^\mu(u)$

We recall that:

a) the basis $\{\tilde{e}_1(u), \tilde{e}_2(u)\}$ is orthonormal: $\langle \tilde{e}_\mu, \tilde{e}_\nu \rangle = \delta_{\mu\nu}$

b) the vector $\xi_{(1)} = \tilde{e}_\mu(u)\tilde{\xi}_{(1)}{}^\mu(u) \in T_u S$ is the image of $\xi_{(0)}$ when the parallel displaced field has been returned to the initial tangent space $T_u S$ of S

c) the region $\Delta\Pi_u$ is the image on S of the infinitesimal parallelogram $\Pi_u\left(\Delta_{(1)}u, \Delta_{(2)}u\right)$ of the parameter-space; it is determined by the point P_u of S and the tangent vectors: $\Delta_{(1)}U = \tilde{e}_\mu(u)\Delta_{(1)}u^\mu$, $\Delta_{(2)}U = \tilde{e}_\mu(u)\Delta_{(2)}u^\mu$

d) $\text{area}\left(\Delta\Pi_u\right) = \sqrt{\tilde{g}(u)}\,\tilde{\omega}^1 \wedge \tilde{\omega}^2\left(\Delta_{(1)}U, \Delta_{(2)}U\right) = \sqrt{g(u)}\,\omega^1 \wedge \omega^2\left(\Delta_{(1)}U, \Delta_{(2)}U\right)$

e) $K(P)$ symbolizes the curvature at the point: $P \equiv P_u \in S$, $u = \left(u^1, u^2\right)$

The connection φ is an isometry (paragraph 11). Hence, the length of any vector field created by the parallel transport of an "initial" vector along a curve C of S is invariant along

C. We verify this statement for the case of the vectors $\xi_{(0)}$ and $\xi_{(1)}$ of 14B.1, by keeping terms up to the first order with respect to the infinitesimal area of the elementary parallelogram.

$$\left\langle \xi_{(1)}, \xi_{(1)} \right\rangle = \left\langle \xi_{(0)}, \xi_{(0)} \right\rangle + \left\langle \tilde{e}_1 \tilde{\xi}_{(0)}{}^1 + \tilde{e}_2 \tilde{\xi}_{(0)}{}^2, -\tilde{e}_1 \tilde{\xi}_{(0)}{}^2 + \tilde{e}_2 \tilde{\xi}_{(0)}{}^1 \right\rangle 2K \,\text{area}(\Delta\Pi_u)$$

$$\left\langle \xi_{(1)}, \xi_{(1)} \right\rangle = \left\langle \xi_{(0)}, \xi_{(0)} \right\rangle$$

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B. We shall now calculate the variation of the angle θ formed by the parallel displaced vector with the vector $\tilde{e}_1(u)$ of the frame field. We name θ_0 the angle of $\xi_{(0)} \in T_u S$ with $\tilde{e}_1(u)$ and θ_1 the angle of $\xi_{(1)} \in T_u S$ with $\tilde{e}_1(u)$ then, the variation $\Delta\theta$ we are looking for equals to the difference:

$$\Delta\theta = \theta_1 - \theta_0$$

Assume that the parallel transferred vector $\xi_{(0)}$ is of unit length. Then, we can write:

$$\xi_{(0)} = \tilde{e}_1(u)\cos\theta_0 + \tilde{e}_2(u)\sin\theta_0$$

$$\xi_{(1)} = \tilde{e}_1(u)\cos(\theta_0 + \Delta\theta) + \tilde{e}_2(u)\sin(\theta_0 + \Delta\theta)$$

By applying 14.14 or 14B.1, we derive the equations:

$$\cos(\theta_0 + \Delta\theta) = \cos\theta_0 - K(u)\sin\theta_0 \,\text{area}(\Delta\Pi_u)$$

$$\sin(\theta_0 + \Delta\theta) = \sin\theta_0 + K(u)\cos\theta_0 \,\text{area}(\Delta\Pi_u)$$

We expand the trigonometric functions appearing in the previous equations in Taylor series and keep terms up to the first order with respect to the variation $\Delta\theta$ of the angle; we obtain the following result:

$$\Delta\theta = K(u)\,\text{area}(\Delta\Pi_u) \qquad (14B.2)$$

By applying 14B.2 we are possible to calculate the change of the angle formed by any vector transported parallel to itself along any closed curve of *S* and returning to its initial position, with the vector $\tilde{e}_1(u)$ of the frame field:

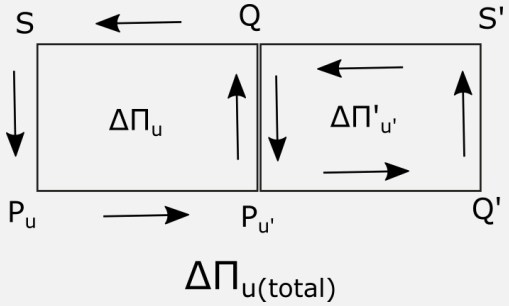

Figure 14.1: Parallel transport of a vector along two successive infinitesimal loops.

Consider the vector field $\xi(v)$ generated by the parallel transport of a vector $\xi_{(0)} \in T_u S$ along the successive, neighboring regions $\Delta\Pi_u$, $\Delta\Pi'_{u'}$ (figure 14.1). We can easily verify (use equation 14.9a) that the images of the initial vector $\xi_0 \in T_u S$ after following either of the following alternative paths are identical:

$$P_u \to P_{u'} \to Q' \to S' \to Q \to S \to P_u$$

$$P_u \to P_{u'} \to Q' \to S' \to Q \to P_{u'} \to Q \to S \to P_u$$

Hence the variation $\Delta\theta$ of the angle between ξ_0 and its final image is given by the equation:

$$\Delta\theta = K(u)\,\mathrm{area}(\Delta\Pi_u) + K(u')\,\mathrm{area}(\Delta\Pi'_{u'}) \qquad (14B.3)$$

Relation 14B.3 can be used to calculate the variation of the angle θ formed by the vector field $\xi(v) = \bar{\varphi}^{(c)}_{v,u}(\xi_{(0)})$ with $\tilde{e}_1(v)$ along any closed curve *c* of the space of the parameters. We only have to approximate the region unclosed by the curve *c* with a collection of infinitesimal parallelograms and apply 14B.3 (figure 14.2). The result is:

$$\theta_{final} - \theta_{initial} = \iint_R K(u)da = \iint_R K(u)\sqrt{\det g(u)}\,\omega^1 \wedge \omega^2(\Delta_{(1)}U, \Delta_{(2)}U) \qquad (14B.4)$$

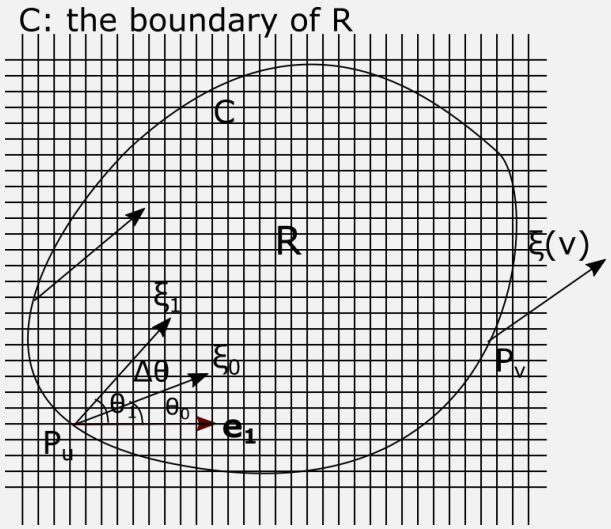

Figure 14.2: Variation of the angle of a parallel displaced field along a closed curve on *S*.

C. *Application and test of 14B.4 for the case of the parallel transport of a vector along a parallel circle on the surface of a sphere S*:

We have already seen that a geometric surface corresponding to a sphere of radius *b* is determined by the metric tensor 14A.1a and the relative basis-field $\{e_1, e_2\}$ obeying the conditions 14A.1b and 14A.2a. The choice of the parameters u^1, u^2 comes from the corresponding parameters u^1, u^2 used for the determination of any point *P* on a spherical surface of center *O* and radius *b* embedded in the 3-dimensional Euclidean space, with respect to an orthonormal coordinate system $Ox^1x^2x^3$ whose principle is at the center of the sphere: u^1 is the polar angle of P related with the axis Ox^3 and u^2 is the projection of the vector ***OP*** on the axis Ox^3 (figure 9.1).

A parallel circle C_z on the sphere *S*, is the image of the curve c_z in the parameters' space, defined by the equations:

$$c_z : \begin{cases} u^1 = q,\ q \in [0, 2\pi] \\ u^2 = z = \text{constant},\ -b < z < b \end{cases}$$

Let us now consider the frame field $\{\tilde{e}_1, \tilde{e}_2\}$ defined by 14A.2c and the vector field:

$$\xi(q,z) = \tilde{e}_\mu(q,z)\tilde{\xi}^\mu(q,z)$$

The vector field $\xi(q,z)$ is generated by the parallel transport along C_z, of the vector:

$$\xi(0,z) \underset{def}{=} \xi_{(0)} = \tilde{e}_2(0,z)$$

The length of a vector field created by the parallel transport of an initial vector along any curve of the geometric surface *S* is invariant (Example 14B-section A); hence the length of $\xi(q,z)$ is unit at any point of the parallel circle C_z:

$$\left|\xi(q,z)\right| = \left|\tilde{e}_2(0,z)\right| = 1$$

The angle formed between $\xi_{(0)}$ and $\tilde{e}_1(0,z)$ is:

$$\theta_{(0)} = \pi / 2$$

We are going to calculate the angle $\theta_{(1)}$ formed by $\xi_{(1)} = \xi(2\pi,z)$ with $\tilde{e}_1(2\pi,z) = \tilde{e}_1(0,z)$ (see relations 14A.2, 3 and "Example 9A"), by following two paths:

a) We solve the equation $D_{\Delta u}\xi(q,z) = 0$ along the curve C_z with initial condition $\xi(0,z) \underset{def}{=} \xi_{(0)} = \tilde{e}_2(0,z)$ and we obtain the analytic form of the field $\xi(q,z)$ explicitly.

b) We apply relation 14B.4

We compare the results obtained by these two alternative methods.

Path a

The vector field $\xi(q,z)$ is transported parallel to itself along the curve C_z. Hence the subsequent relations are true:

$$D_{\Delta u}\xi(q,z) = D_{\Delta u}\left(\tilde{e}_\mu \tilde{\xi}^\mu\right) = 0,\ \Delta u = (\Delta q, 0),\ \Delta q \to 0$$

$$\delta_{\mu\nu} D_{\Delta u}\tilde{\xi}^\mu + \left\langle \tilde{e}_\nu, D_{\Delta u}\tilde{e}_\mu \right\rangle \tilde{\xi}^\mu = 0$$

$$\frac{d\tilde{\xi}^\nu}{dq} + \tilde{\tilde{\Gamma}}_{\nu\mu\kappa}\tilde{\xi}^\mu \frac{du^\kappa}{dq} = 0$$

$$\frac{du^1}{dq} = 1$$

$$\frac{du^2}{dq} = 0$$

The symbols $\tilde{\tilde{\Gamma}}_{\nu\mu\kappa}$ have been calculated in 14A.4; we substitute in the previous system and we are led to the differential equations:

$$\begin{aligned} \frac{d\tilde{\xi}^1}{dq} - \frac{z}{b}\tilde{\xi}^2 &= 0 \\ \frac{d\tilde{\xi}^2}{dq} + \frac{z}{b}\tilde{\xi}^1 &= 0 \end{aligned} \qquad (14B.5a)$$

The solution of 14B.5a with initial condition $\xi(0,z) \underset{def}{=} \xi_{(0)} = \tilde{e}_2(0,z)$ is given by the functions:

$$\begin{aligned} \tilde{\xi}^1 &= \sin\left(\frac{z}{b}q\right) \\ \tilde{\xi}^2 &= \cos\left(\frac{z}{b}q\right) \end{aligned} \qquad (14B.5b)$$

According to 14A.3, for $q = 2\pi$ the vectors of the frame field are identical to the initial vectors of the frame field, at q=0:

$$\tilde{e}_\mu(0,z) = \tilde{e}_\mu(2\pi,z)$$

For $q = 2\pi$ the vector field $\xi(q,z)$ returns to its original tangent space:

$$\xi_{(1)} \underset{def}{=} \xi(2\pi,z) \in T_{(2\pi,z)}S \equiv T_{(0,z)}S \qquad (14B.6)$$

By 14B.5b we result that at the end of its trip along the parallel circle C_z, the field $\xi(q,z)$ takes the form:

$$\xi_{(1)} = \tilde{e}_1(0,z)\sin\left(\frac{2\pi z}{b}\right) + \tilde{e}_2(0,z)\cos\left(\frac{2\pi z}{b}\right) \qquad (14B.7)$$

From 14B.7 we can easily calculate the angle $\theta_{(1)}$ formed between $\xi_{(1)}$ and $\tilde{e}_1(0,z)$

$$\cos\theta_{(1)} = \frac{\langle \tilde{e}_1(0,z), \xi_{(1)} \rangle}{|\tilde{e}_1(0,z)||\xi_{(1)}|} = \sin\left(\frac{2\pi z}{b}\right)$$

$$\theta_{(1)} = \frac{\pi}{2} - \frac{2\pi z}{b}$$

We conclude that the angle between $\xi_{(1)}$ and $\xi_{(0)}$ is:

$$\Delta\theta = \theta_{(1)} - \theta_{(0)} = -\frac{2\pi z}{b} \qquad (14B.8)$$

Path b

Now, we are going to calculate $\Delta\theta = \theta_{(1)} - \theta_{(0)}$ by applying 14B.4:

$$\Delta\theta = \theta_{(1)} - \theta_{(0)} = \iint_R K(u)da = \iint_R K(u)\sqrt{\det g(u)}\,\omega^1 \wedge \omega^2(\Delta_{(1)}U, \Delta_{(2)}U)$$

$C_z = \partial R$, $\det g(u) = b^2$, $\det \tilde{g}(u) = 1$

$$\Delta_{(1)}U = e_1(u)\Delta u^1 = \tilde{e}_1(u)\sqrt{b^2 - (u^2)^2}\,\Delta u^1$$

$$\Delta_{(2)}U = e_2(u)\Delta u^2 = \tilde{e}_2(u)\frac{b}{\sqrt{b^2 - (u^2)^2}}\Delta u^2$$

$\omega^1(\Delta_{(1)}U) = \Delta u^1$, $\omega^1(\Delta_{(2)}U) = 0$, $\omega^2(\Delta_{(1)}U) = 0$, $\omega^2(\Delta_{(2)}U) = \Delta u^2$

$\tilde{\omega}^1(\Delta_{(1)}U) = \sqrt{b^2 - (u^2)^2}\,\Delta u^1$, $\tilde{\omega}^1(\Delta_{(2)}U) = 0, \tilde{\omega}^2(\Delta_{(1)}U) = 0$, $\tilde{\omega}^2(\Delta_{(2)}U) = \frac{b}{\sqrt{b^2 - (u^2)^2}}\Delta u^2$

$$\text{area}(\Delta\Pi_u) = \sqrt{\tilde{g}(u)}\,\tilde{\omega}^1 \wedge \tilde{\omega}^2(\Delta_{(1)}U, \Delta_{(2)}U) = \sqrt{g(u)}\,\omega^1 \wedge \omega^2(\Delta_{(1)}U, \Delta_{(2)}U) = b\Delta u^1\Delta u^2$$

Hence:

$$\Delta\theta = \theta_{(1)} - \theta_{(0)} = \frac{1}{b}\int_0^{2\pi} du^1 \int_z^b du^2 - 2\pi = 2\pi\left(1 - \frac{z}{b}\right) - 2\pi = -\frac{2\pi z}{b}$$

This result is identical to 14B.8.

15. **Geodesic curvature**

In the first section of the present paragraph we are going to define the tangent and the normal vector field on a curve lying on a geometric surface *S*. From the relation of these vector fields and their variation along the curve, the notion of the geodesic curvature, along with the Frenet-Serret equations of the curve, is emerged.

Let us consider a certain curve $c(s) = (c^1(s), c^2(s))$ lying in the parameter space of the geometric surface *S* and its image-curve $C : s \to P_{c(s)} \in S$ on *S*; the parameter *s* is the length of the image-curve *C*:

$$s = \int_0^s \left(g_{\mu\nu}(\sigma)dc^\mu(\sigma)dc^\nu(\sigma)\right)^{1/2}$$

The coordinates of the tangent vector $\dot{c}(s) = (\dot{c}^1(s), \dot{c}^2(s))$ of *c* satisfy the condition:

$$g_{\mu\nu}(s)\dot{c}^\mu(s)\dot{c}^\nu(s) = 1$$

As usually, we symbolize:

$$\dot{c}^{\mu}(s) \underset{def}{=} \frac{dc^{\mu}(s)}{ds}$$

We define the vector field $T(s)$ which is tangent to the curve C, at its points $P_{c(s)}$ and its length is unit for any value of s:

$$T(s) = e_{\mu}\left(c(s)\right)\dot{c}^{\mu}(s) \in T_{c(s)}S$$

$$\left\langle T(s), T(s)\right\rangle = 1$$

The variation of $T(s)$ along C is calculated by its covariant differential:

$$D_{\Delta c}T(s) = e_{\nu}(c)\left(\ddot{c}^{\nu} + \bar{\Gamma}^{\nu}_{\mu\kappa}(c)\dot{c}^{\mu}\dot{c}^{\kappa}\right)\Delta s,\ \Delta s \to 0 \tag{15.1}$$

Given that $T(s)$ is of unit length, we imply that:

$$\left\langle D_{\Delta c}T(s), T(s)\right\rangle = 0 \tag{15.2}$$

Hence the covariant derivative of $T(s)$ is a vector in the tangent space $T_{c(s)}S$ normal to the tangent vector $T(s)$ of C:

$$\frac{D_{dc}T(s)}{ds} \underset{def}{=} \lim_{\Delta s\to 0}\frac{D_{\Delta c}T(s)}{\Delta s} = e_{\nu}(c)\left(\ddot{c}^{\nu} + \bar{\Gamma}^{\nu}_{\mu\kappa}(c)\dot{c}^{\mu}\dot{c}^{\kappa}\right)$$

Let $k_g(s)$ be a differentiable real function of s and $N(s)$ a unit vector field $N(s) \in T_{c(s)}S$ such that:

$$\frac{D_{dc}T(s)}{ds} = k_g(s)N(s) \tag{15.3}$$

The real function $k_g(s)$ is called "the **geodesic curvature** of C" and $N(s)$ "the unit **normal** of C on S".

Remark: Notice that there is an ambiguity about the definition of $k_g(s)$ and $N(s)$ having to do with the choice of the orientation of $N(s)$. We treat this problem as follows:

Consider the frame field $\left\{\tilde{e}_1(s), \tilde{e}_2(s)\right\}$ defined in paragraph 14. In each tangent space $T_{c(s)}S$ the bilinear antisymmetric form $\tilde{\omega}^1 \wedge \tilde{\omega}^2$ is defined (see paragraph 8):

$$\tilde{\omega}^1\left(\tilde{e}_1(s)\right) = 1,\ \tilde{\omega}^1\left(\tilde{e}_2(s)\right) = 0,\ \tilde{\omega}^2\left(\tilde{e}_1(s)\right) = 0,\ \tilde{\omega}^2\left(\tilde{e}_2(s)\right) = 1$$

$$\tilde{\omega}^1 \wedge \tilde{\omega}^2\left(\tilde{e}_1(s), \tilde{e}_2(s)\right) = -\tilde{\omega}^1 \wedge \tilde{\omega}^2\left(\tilde{e}_2(s), \tilde{e}_1(s)\right) = 1$$

Given that $T(s)$ is of unit length, we write:

$$T(s) = \tilde{e}_1(s)\cos\theta + \tilde{e}_2(s)\sin\theta$$

The angle θ is formed by $\tilde{e}_1(s)$ with $T(s)$.

$N(s)$ is of unit length and orthogonal to $T(s)$. Hence it might have one of the forms:

$$N(s) = -\tilde{e}_1(s)\sin\theta + \tilde{e}_2(s)\cos\theta$$

Or:

$$N'(s) = \tilde{e}_1(s)\sin\theta - \tilde{e}_2(s)\cos\theta$$

We choose the orientation of $N(s)$ so that the following equation to be satisfied:

$$\tilde{\omega}^1 \wedge \tilde{\omega}^2\left(T(s), N(s)\right) = +1$$

Hence $N(s)$ is determined by the expression:

$$N(s) = -\tilde{e}_1(s)\sin\theta + \tilde{e}_2(s)\cos\theta$$

•

In order to simplify the symbolism, we introduce the symbol D_c meaning the covariant derivative of a vector field along the curve C. We write:

$$D_cT(s) \underset{def}{=} \frac{D_{dc}T(s)}{ds} = \lim_{\Delta s\to 0}\frac{D_{\Delta c}T(s)}{\Delta s} = e_{\nu}(c)\left(\ddot{c}^{\nu} + \bar{\Gamma}^{\nu}_{\mu\kappa}(c)\dot{c}^{\mu}\dot{c}^{\kappa}\right)$$

We summarize the following identities:

$$D_c T(s) = k_g(s) N(s)$$
$$\langle D_c T(s), N(s) \rangle = k_g(s)$$
$$\langle D_c T(s), N(s) \rangle + \langle T(s), D_c N(s) \rangle = 0$$
$$\left(k_g(s)\right)^2 = \langle D_c T(s), D_c T(s) \rangle = g_{\nu\lambda}\left(\ddot{c}^\nu + \bar{\Gamma}^\nu_{\mu\kappa}(c)\dot{c}^\mu\dot{c}^\kappa\right)\left(\ddot{c}^\lambda + \bar{\Gamma}^\lambda_{\rho\sigma}(c)\dot{c}^\rho\dot{c}^\sigma\right)$$
$$D_c N(s) = -k_g(s) T(s)$$

If the curve C is a geodesic on S, the covariant derivative of $T(s)$ is zero for every s. Then, from 15.3 we infer that its geodesic curvature k_g equals to zero for any value of s, and vice versa; if $k_g(s)=0$ along C, then C is a geodesic of S.

Variation of the angles formed by the tangent of a curve on a geometric surface and a certain frame field

When we are moving along a curve C of S, the angle θ formed by the vector field $T(s)$ and the element $\tilde{e}_1(s)$ of the frame field $\{\tilde{e}_1(s), \tilde{e}_2(s)\}$ defined in paragraph 14, is changing. The variation of θ is caused by two factors: a) the analytic expression of the curve, b) the connection defined on the geometric surface. In this section of the present paragraph, we derive an analytic formula that affords us to calculate the variation of θ if we know the geodesic curvature of the curve as a function of its length s and the connection forms corresponding to the frame field $\{\tilde{e}_1(s), \tilde{e}_2(s)\}$ of the geometric surface.

Consider again the curve C on S which is the image of the curve c in the parameters' space. The parameter s of the curve c is the length of C measured from some point P_0 of C. We symbolize:

$$c(s) = \left(c^1(s), c^2(s)\right),\ \Delta c = \left(\Delta c^1, \Delta c^2\right),\ \Delta c^\kappa = \dot{c}^\kappa(s)\Delta s$$
$$\Delta C = e_\mu\left(c(s)\right)\Delta c^\mu = e_\mu\left(c(s)\right)\dot{c}^\mu(s)\Delta s$$

Starting from 15.3 and using 14.7, the subsequent relationships arise:

$$D_{\Delta c} T(s) = k_g(s) N(s) \Delta s$$
$$D_{\Delta c}\left(\tilde{e}_1(s)\cos\theta + \tilde{e}_2(s)\sin\theta\right) = \Delta s\, k_g(s)\left(-\tilde{e}_1(s)\sin\theta + \tilde{e}_2(s)\cos\theta\right)$$
$$D_{\Delta c}\tilde{e}_1\cos\theta + \tilde{e}_1 D_{\Delta c}\cos\theta + D_{\Delta c}\tilde{e}_2\sin\theta + \tilde{e}_2 D_{\Delta c}\sin\theta = \Delta s\, k_g(s)\left(-\tilde{e}_1\sin\theta + \tilde{e}_2\cos\theta\right)$$
$$\tilde{e}_1\left(-\sin\theta\,\Delta\theta + \tilde{\omega}_{12}(\Delta C)\sin\theta\right) + \tilde{e}_2\left(\cos\theta\,\Delta\theta - \tilde{\omega}_{12}(\Delta C)\cos\theta\right) = -\tilde{e}_1\Delta s\, k_g\sin\theta + \tilde{e}_2\Delta s\, k_g\cos\theta$$
$$\Delta\theta = k_g\Delta s + \tilde{\omega}_{12}(\Delta C)$$

We conclude that the variation $\Delta\theta$ of the angle θ formed by $T(s)$ with $\tilde{e}_1(s)$ when moving along the curve C from its point $C(s)$ to the neighboring one $C(s+\Delta s)$, $\Delta s \to 0$ is calculated by the equation:

$$\Delta\theta = k_g\Delta s + \tilde{\omega}_{12}(\Delta C) \qquad (15.4a)$$

The connection forms $\tilde{\omega}_{\mu\nu}$ have been defined in paragraph 14; they are related to the frame field $\{\tilde{e}_1(s), \tilde{e}_2(s)\}$ (see relations 14.6a-d):

$$\tilde{\omega}_{11}(\Delta C) = 0 \quad \tilde{\omega}_{22}(\Delta C) = 0$$
$$\tilde{\omega}_{12}(\Delta C) = \tilde{\bar{\Gamma}}_{12\kappa}\Delta c^\kappa = -\tilde{\bar{\Gamma}}_{21\kappa}\Delta c^\kappa = -\tilde{\omega}_{21}(\Delta C)$$

According to 15.4a, if we move on C from a point $C(s_1)$ to another $C(s_2)$, the variation of the angle θ is calculated by the equation:

$$\theta_2 - \theta_1 = \int_{C(s_1)}^{C(s_2)} k_g(s)ds + \int_{C(s_1)}^{C(s_2)} \tilde{\omega}_{12}\left(\dot{c}(s)ds\right) \qquad (15.4b)$$

For the case that C is a geodesic, the previous relation is reduced to the following

$$\theta_2 - \theta_1 = \int_{C(s_1)}^{C(s_2)} \tilde{\omega}_{12}\left(\dot{c}(s)ds\right) \qquad (15.4c)$$

An expression of the curvature in the language of the connection forms

In this section we relate the variation of the angle θ formed by a parallel transported vector field along an infinitesimal loop with the element $\tilde{e}_1(u)$ of the frame field $\{\tilde{e}_1(u), \tilde{e}_2(u)\}$ defined in paragraph 14. We derive a relation of the total variation of this angle with the connection forms and another one involving the curvature of the surface. From the combination of these relations, the intimate relationship of the connection forms with the curvature of the geometric surface is emerged, usually called "the second structural equation" (2).

Consider the frame field $\{\tilde{e}_1(u), \tilde{e}_2(u)\}$ and the connection on S determined by the Christoffel symbols $\tilde{\tilde{\Gamma}}_{\nu\mu\kappa}$ with respect to the considered frame field (see paragraph 14). Assume again, an infinitesimal orthogonal parallelogram $\Delta\pi_u\left(\Delta_{(1)}u, \Delta_{(2)}u\right)$ of the parameters' u-space with vertices the points:

$u = \left(u^1, u^2\right)$, $u + \Delta_{(1)}u$, $u + \Delta_{(1)}u + \Delta_{(2)}u$, $u + \Delta_{(2)}u$

$\Delta_{(1)}u = \left(\Delta u^1, 0\right)$, $\Delta_{(2)}u = \left(0, \Delta u^2\right)$, $\Delta u^1 \to 0$, $\Delta u^2 \to 0$

The parallelogram $\Delta\pi_u\left(\Delta_{(1)}u, \Delta_{(2)}u\right)$ is mapped to the closed region $\Delta\Pi_u$ of S.

Let us now create the vector field $\xi(v)$ by the parallel transport of a vector $\xi_{(0)}(u) \in T_uS$ along the boundary of the infinitesimal parallelogram; we write:

$\xi(v) = \tilde{e}_\mu(v)\tilde{\xi}^\mu(v)$, $v \in \partial\Delta\pi_u\left(\Delta_{(1)}u, \Delta_{(2)}u\right)$

$\xi_{(0)}(u) = \tilde{e}_1(u)\cos\theta_{(0)} + \tilde{e}_2(u)\sin\theta_{(0)} \in T_uS$

The vector field $\xi(v)$ is of unit length (see paragraph 11); hence:

$\tilde{\xi}^1(v) = \cos\theta(v)$, $\tilde{\xi}^2(v) = \sin\theta(v)$

The angle $\theta(v)$ is formed between $\xi(v)$ and the element $\tilde{e}_1(v)$ of the frame field.

Let $\xi_{(1)}(u) = \tilde{e}_1(u)\cos\theta_{(1)} + \tilde{e}_2(u)\sin\theta_{(1)} \in T_uS$ be the form of the field $\xi(v)$ when it returns to the tangent space T_uS from which it has been departed. According to equation 14.14, the variation of the components of the vectors $\xi_{(0)}(u)$ and $\xi_{(1)}(u)$ is calculated by the relations:

$\cos(\theta_{(0)} + \Delta\theta) = \cos\theta_{(0)} - K(u)\,\text{area}(\Delta\Pi_u)\sin\theta_{(0)}$

$\sin(\theta_{(0)} + \Delta\theta) = \sin\theta_{(0)} + K(u)\,\text{area}(\Delta\Pi_u)\cos\theta_{(0)}$

We have put $\theta_{(1)} = \theta_{(0)} + \Delta\theta$ and have kept terms up to the first order with respect to the area of the infinitesimal region. We imply the equation:

$$\Delta\theta = K(u)\,\text{area}(\Delta\Pi_u) \qquad (15.5)$$

We now express the $\text{area}(\Delta\Pi_u)$ with respect to the differential 1-forms $\tilde{\omega}^\mu$ determined by the relations (paragraph 10 and Example 14B, part C):

$\tilde{\omega}^{\mu}\left(\xi(v)\right)=\tilde{\omega}^{\mu}\left(\tilde{e}_{\kappa}(v)\tilde{\xi}^{\kappa}(v)\right)=\tilde{\xi}^{\mu}(v)$

Define:

$\Delta_{(1)}U=e_1(u)\Delta u^1=\tilde{e}_{\rho}(u)\tilde{L}^{\rho}_1\Delta u^1$

$\Delta_{(2)}U=e_2(u)\Delta u^2=\tilde{e}_{\rho}(u)\tilde{L}^{\rho}_2\Delta u^2$

According to the remarks of paragraph 14, we get:

$\det\tilde{L}(u)=\dfrac{1}{\det L(u)}=\sqrt{\det g(u)}$

$\text{area}(\Delta\Pi_u)=\sqrt{\det g}\,\omega^1\wedge\omega^2\left(\Delta_{(1)}U,\Delta_{(2)}U\right)=\det\tilde{L}\,\Delta u^1\Delta u^2$

$\tilde{\omega}^1\wedge\tilde{\omega}^2\left(\Delta_{(1)}U,\Delta_{(2)}U\right)=\tilde{\omega}^1\wedge\tilde{\omega}^2\left(\tilde{e}_{\rho}\tilde{L}^{\rho}_1\Delta u^1,\tilde{e}_{\rho}\tilde{L}^{\rho}_2\Delta u^2\right)=$

$=\tilde{L}^1_1\Delta u^1\tilde{L}^2_2\Delta u^2-\tilde{L}^1_2\Delta u^2\tilde{L}^2_1\Delta u^1=\det\tilde{L}\,\Delta u^1\Delta u^2=\text{area}(\Delta\Pi_u)$

Hence:

$$\text{area}(\Delta\Pi_u)=\tilde{\omega}^1\wedge\tilde{\omega}^2\left(\Delta_{(1)}U,\Delta_{(2)}U\right)\qquad(15.5a)$$

From 15.5 and 15.5a we infer that:

$$\Delta\theta=K(u)\tilde{\omega}^1\wedge\tilde{\omega}^2\left(\Delta_{(1)}U,\Delta_{(2)}U\right)\qquad(15.6)$$

On the other hand, given that $\xi(v)$ is been transported parallel to itself along $\partial\Delta\Pi_u$ the variation $\delta\tilde{\xi}^{\mu}(v)$ of its components satisfy the equation:

$\delta\tilde{\xi}^{\mu}(v)=-\tilde{g}^{\mu\rho}\tilde{\tilde{\Gamma}}_{\rho\lambda\kappa}(v)\tilde{\xi}^{\lambda}(v)\,\delta u^{\kappa}=-\tilde{\tilde{\Gamma}}_{\mu\lambda\kappa}(v)\tilde{\xi}^{\lambda}(v)\,\delta u^{\kappa}$

Where:

(a) The couple $\delta u=\left(\delta u^1,\delta u^2\right)$ determines an infinitesimal transport from the point v to the point $v+\delta u$ along the boundary $\partial\Delta\Pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)$ in the parameter-space.

(b) The vector $\delta U=e_{\mu}(v)\delta u^{\mu}\in T_vS$ corresponds to the infinitesimal transport $v\to v+\delta u$ along $\partial\Delta\Pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)$ in the space of the parameters.

(c) The vector field $\xi(v)$ is of unit length and we can write:

$\tilde{\xi}^1(v)=\cos\theta(v)$, $\tilde{\xi}^2(v)=\sin\theta(v)$

(d) The action of the 1-form $\tilde{\omega}_{12}$ at $\delta U=e_{\mu}(v)\delta u^{\mu}\in T_vS$ returns (relations 14.6a,b and c):

$\tilde{\omega}_{12}(\Delta U)=\tilde{\tilde{\Gamma}}_{12\kappa}(u)\Delta u^{\kappa}=\tilde{\tilde{\Gamma}}_{12\kappa}(u)\omega^{\kappa}(\Delta U)$

The external derivative of $\tilde{\omega}_{12}$ is the 2-form (relation 8.9):

$d\tilde{\omega}_{12}\left(\Delta_{(1)}U,\Delta_{(2)}U\right)\underset{def}{=}d_{\Delta_{(1)}U}\tilde{\omega}_{12}(\Delta_{(2)}U)=\partial_{\lambda}\tilde{\tilde{\Gamma}}_{12\kappa}(u)\Delta_{(1)}u^{\lambda}\wedge\Delta_{(2)}u^{\kappa}$

By using 14.5b, 14.6b and applying the Stokes' theorem (relation 8.10a) we derive the consequent relationships:

$\delta\tilde{\xi}^1=-\tilde{\tilde{\Gamma}}_{12\kappa}\tilde{\xi}^2\delta u^{\kappa}=-\tilde{\omega}_{12}(\delta U)\tilde{\xi}^2$

$-\sin\theta\,\delta\theta=-\tilde{\omega}_{12}(\delta U)\sin\theta$

$\delta\theta=\tilde{\omega}_{12}(\delta U)$

$\Delta\theta=\theta_{(1)}-\theta_{(0)}=\oint\limits_{\partial\Delta\Pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)}\tilde{\omega}_{12}(\delta U)=\iint\limits_{\Delta\Pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)}d\tilde{\omega}_{12}\approx d\tilde{\omega}_{12}\left(\Delta_{(1)}U,\Delta_{(2)}U\right)$

As usually, terms up to the first order with respect to the area of $\Delta\Pi_u\left(\Delta_{(1)}u,\Delta_{(2)}u\right)$ have been kept. Hence we end up to the relationship:

$$\Delta\theta = d\tilde{\omega}_{12}\left(\Delta_{(1)}U, \Delta_{(2)}U\right) \tag{15.7}$$

We now combine 15.6 and 7 and we derive the identity: [2]

$$d\tilde{\omega}_{12}\left(\Delta_{(1)}U, \Delta_{(2)}U\right) = K(u)\,\tilde{\omega}^1 \wedge \tilde{\omega}^2\left(\Delta_{(1)}U, \Delta_{(2)}U\right) \tag{15.8a}$$

$$d\tilde{\omega}_{12} = K(u)\,\tilde{\omega}^1 \wedge \tilde{\omega}^2 \tag{15.8b}$$

The derived equation 15.8b relates the connection forms with the curvature of the geometric surface; it is known as the "second structural equation" of the geometric surface [2].

Example 15A

Application of the second structural equation for case of a sphere

In this example, by applying relation 15.8, we are going to confirm once more, that the curvature of the sphere *S* defined in the Example 14A, is given by the relation:

$K(u) = 1/b^2$

The implementation of this objective requires the evaluation of the 2-form $\tilde{\omega}^1 \wedge \tilde{\omega}^2$ and of the external derivative of the 1-form $\tilde{\omega}_{12}$ for the case of the sphere *S*. Then, by substitution of the achieved analytic expressions in 15.8, we expect to obtain the correct result for the curvature of *S* at anyone of its points.

a) According to the definitions of the 1-forms $\tilde{\omega}^1$, $\tilde{\omega}^2$ (see paragraph 15-geodesic curvature) we have:

$$\tilde{\omega}^1 \wedge \tilde{\omega}^2\left(\Delta_{(1)}U, \Delta_{(2)}U\right) = \tilde{\omega}^1 \wedge \tilde{\omega}^2\left(e_\mu \Delta_{(1)}u^\mu, e_\nu \Delta_{(2)}u^\nu\right) = \tilde{\omega}^1 \wedge \tilde{\omega}^2\left(\tilde{e}_\lambda \tilde{L}^\lambda_\mu \Delta_{(1)}u^\mu, \tilde{e}_\rho \tilde{L}^\rho_\nu \Delta_{(2)}u^\nu\right) =$$

$$= \tilde{\omega}^1\left(\tilde{e}_\lambda \tilde{L}^\lambda_\mu \Delta_{(1)}u^\mu\right)\tilde{\omega}^2\left(\tilde{e}_\rho \tilde{L}^\rho_\nu \Delta_{(2)}u^\nu\right) - \tilde{\omega}^1\left(\tilde{e}_\rho \tilde{L}^\rho_\nu \Delta_{(2)}u^\nu\right)\tilde{\omega}^2\left(\tilde{e}_\lambda \tilde{L}^\lambda_\mu \Delta_{(1)}u^\mu\right) =$$

$$= \tilde{L}^1_\mu \Delta_{(1)}u^\mu \tilde{L}^2_\nu \Delta_{(2)}u^\nu - \tilde{L}^1_\nu \Delta_{(2)}u^\nu \tilde{L}^2_\mu \Delta_{(1)}u^\mu = \tilde{L}^1_\mu \tilde{L}^2_\nu\left(\Delta_{(1)}u^\mu \Delta_{(2)}u^\nu - \Delta_{(2)}u^\mu \Delta_{(1)}u^\nu\right) =$$

$$= \tilde{L}^1_\mu \tilde{L}^2_\nu\, \omega^\mu \wedge \omega^\nu\left(\Delta_{(1)}U, \Delta_{(2)}U\right) = \left(\tilde{L}^1_1 \tilde{L}^2_2 - \tilde{L}^1_2 \tilde{L}^2_1\right)\omega^1 \wedge \omega^2\left(\Delta_{(1)}U, \Delta_{(2)}U\right)$$

Hence:

$$\tilde{\omega}^1 \wedge \tilde{\omega}^2 = \frac{1}{\det L}\,\omega^1 \wedge \omega^2 \tag{15A.1}$$

According to 14A.2e of the example 14A, we have:

$\det L = 1/b$

$$\tilde{\omega}^1 \wedge \tilde{\omega}^2 = b\,\omega^1 \wedge \omega^2 \tag{15A.2}$$

b) From 15.8b and 14A.4 (see example 14A), we have

$$\tilde{\omega}_{12} = \tilde{\tilde{\Gamma}}_{12\kappa}(u)\omega^\kappa = -\frac{u^2}{b}\omega^1$$

$$d\tilde{\omega}_{12} = \partial_\lambda\left(-\frac{u^2}{b}\right)\omega^\lambda \wedge \omega^1 = \partial_2\left(-\frac{u^2}{b}\right)\omega^2 \wedge \omega^1 = \frac{1}{b}\omega^1 \wedge \omega^2$$

By taking into account 15A.2, we obtain:

$$d\tilde{\omega}_{12} = \frac{1}{b^2}\tilde{\omega}^1 \wedge \tilde{\omega}^2$$

We conclude that:

$K(u) = 1/b^2$

This is the anticipated result.

The Gauss-Bonnet theorem offers a way to calculate the sum of the angles of a polygon on the geometric surface *S*, whose sides are segments of geodetics of *S*. Such a polygon will be called "**geodesic polygon**". The formulation of the problem and the outlined proof of the stated theorem is an immediate result of the preceding sections of paragraph 15. The most interesting notification pointed by this theorem is the dependence of the sum of the angles of the geodesic polygon on the curvature of the geometric surface. In the case of a Euclidean plane, where the curvature is zero, the theorem is reduced to the well-known from the Euclidean Geometry relation:

$$\sum_{j=1}^{N} \varphi_j = (N-2)\pi$$

N (*N*>2) is the number of the sides and $\varphi_j \; j = 1,2...N$ are the internal angles of the polygon.

We shall formulate this theorem for the case of a geodetic triangle of *S* and we leave the generalization as an exercise for the interested reader.
Consider the triangle $P_1P_2P_3$ of *S* (figure 15.1), where the points P_1, P_2, P_3 are images of the points $u_{(1)}$, $u_{(2)}$, $u_{(3)}$ of the parameter-space:

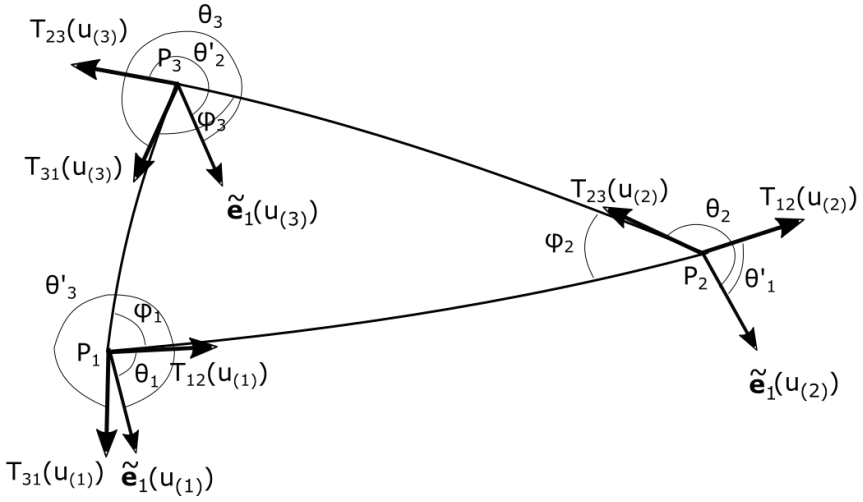

Figure 15.1: A geodesic triangle on a geometric surface *S*. At each vertex P_j, the tangent vectors of the intersecting sides are related by a rotation transformation with angle: $\pi - \varphi_j$

We symbolize φ_j the internal angle of the triangle corresponding to the vertex P_j.

$$u_{(1)} = \left(u^1_{(1)}, u^2_{(1)}\right) \to P_1$$

$$u_{(2)} = \left(u^1_{(2)}, u^2_{(2)}\right) \to P_2$$

$$u_{(3)} = \left(u^1_{(3)}, u^2_{(3)}\right) \to P_3$$

The sides P_1P_2, P_2P_3, P_3P_1 are segments of geodesic curves of *S*. The corresponding tangent vectors of unit length are symbolized by $T_{12}(u)$, $P_u \in P_1P_2$ and so on, as is illustrated in figure 15.1.

Now, imagine that we are moving together with the frame field $\{\tilde{e}_1(u), \tilde{e}_1(u)\}$ along the sides of the triangle, starting from the vertex P_1: from P_1 we go to P_2, then to P_3 and back to P_1. As we are moving on the perimeter of the triangle, let us watch the changes of the tangent vector at each side:

We have named φ_j the internal angle of the triangle at the corresponding vertex P_j j=1,2,3.

At every vertex P_j of the triangle, the tangent vector has to be rotated at an angle $\pi - \varphi_j$ in order to become tangent to the next side.

Let us name $\theta(u)$ the angle formed by the tangent vector $T_{12}(u)$ with the element $\tilde{e}_1(u) \in T_u S$ of the frame field, at any point P_u of our journey. At the end of the trip the tangent vector comes to its original position: it is identical to the initial tangent vector $T_{12}(u_{(1)})$. Hence, the total change of the angle $\theta(u)$ equals to 2π (figure 15.1).

We symbolize:

a) $\theta_1 \underset{def}{=} \theta\left(u_{(1)}\right)$ the initial angle of the $T_{12}(u_{(1)})$ with $\tilde{e}_1(u)$ when we depart our journey,

b) $\theta_{1final} \underset{def}{=} \theta_{final}\left(u_{(1)}\right)$ the final angle of the same vectors after the journey along the triangle perimeter has been completed:

These two angles differ by 2π rad i.e.:

$\theta_{1final} - \theta_1 = 2\pi$

At the same time, the total change $\theta_{1final} - \theta_1$ of the angle $\theta(u)$ is calculated from the expression (see figure 15.1):

$\theta_{1final} - \theta_1 = \theta_1' - \theta_1 + \pi - \varphi_2 + \theta_2' - \theta_2 + \pi - \varphi_3 + \theta_3' - \theta_3 + \pi - \varphi_1$

We conclude that:

$2\pi = (\theta_1' - \theta_1) + (\theta_2' - \theta_2) + (\theta_3' - \theta_3) - (\varphi_1 + \varphi_2 + \varphi_3) + 3\pi$

$\varphi_1 + \varphi_2 + \varphi_3 = \pi + (\theta_1' - \theta_1) + (\theta_2' - \theta_2) + (\theta_3' - \theta_3)$

Given that the sides of the triangle are geodesic segments, we apply equation 15.4c and the previous expression takes the form:

$$\varphi_1 + \varphi_2 + \varphi_3 = \pi + \oint_{\partial(P_1P_2P_3)} \tilde{\omega}_{12}\left(T(s)\Delta s\right) \tag{15.9a}$$

The integral at the right hand side of 15.9a has to be calculated along the boundary of the geodesic triangle $P_1P_2P_3$.

By applying the Stokes' theorem (relation 8.10b) 15.9a is transformed to the following:

$$\varphi_1 + \varphi_2 + \varphi_3 = \pi + \iint_{(P_1P_2P_3)} d\tilde{\omega}_{12}\left(\Delta_{(1)}U, \Delta_{(2)}U\right) \tag{15.9b}$$

$\Delta_{(1)}U = e_1(u)\Delta u^1 \in T_u S$, $\Delta_{(2)}U = e_2(u)\Delta u^2 \in T_u S$, $P_u \in \left(P_1P_2P_3\right)$

The points P_u belong to the geodesic triangle.

The infinitesimal parallelograms $\pi_u\left(\Delta_{(1)}u, \Delta_{(2)}u\right)$, $\Delta_{(1)}u = \left(\Delta u^1, 0\right)$, $\Delta_{(2)}u = \left(0, \Delta u^2\right)$ specify an appropriate partition on the parameters' space that, in turn, determines a partition of the geodesic triangle on S, appropriate to carry out the calculation of the integral (see for example, paragraph 14).

We proceed by using 15.8b; we find that:

$$\varphi_1 + \varphi_2 + \varphi_3 = \pi + \iint_{(P_1P_2P_3)} K(u)\,\tilde{\omega}_1 \wedge \tilde{\omega}_2\left(\Delta_{(1)}U, \Delta_{(2)}U\right) \tag{15.9c}$$

$K(u)$ is the curvature of S at its point P_u of the geodesic triangle.

Finally, by using 15.5a, equation 15.9c takes the form:

$$\varphi_1 + \varphi_2 + \varphi_3 = \pi + \iint_{(P_1P_2P_3)} K(u)\,\mathrm{area}\left(\Delta\Pi_u\right) \qquad (15.9d)$$

As usually (see paragraph 12), $\Delta\Pi_u$ symbolize the image of the infinitesimal parallelogram $\pi_u\left(\Delta_{(1)}u, \Delta_{(2)}u\right)$ of the parameter-space, on the geometric surface *S*. From 15.9d, it is obvious that if our surface is flat i.e. if its curvature is zero everywhere, then the sum of the angles equals π rad, which is the case of a Euclidean plane.

Example 15B

Application of the Gauss-Bonnet theorem for the case of a geodesic triangle on a sphere

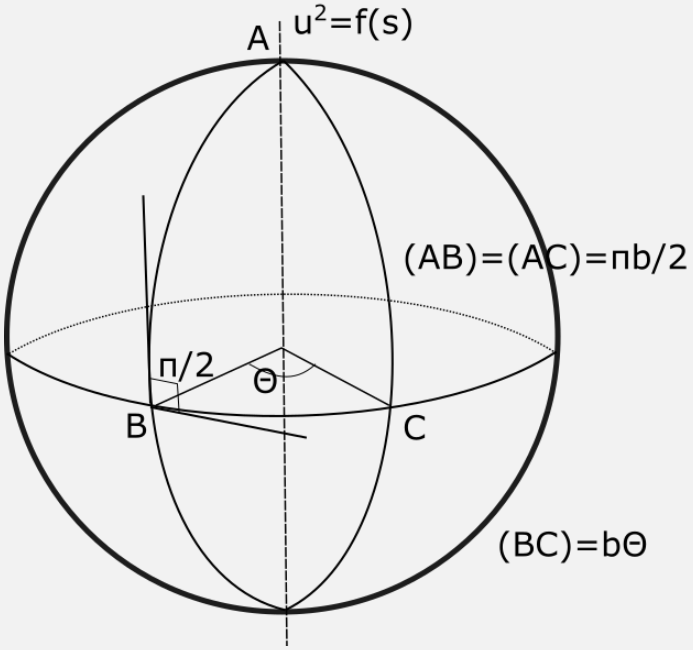

Figure 15.2: A geodesic triangle *ABC* on a spherical surface. The sum of its internal angles is: $\hat{A} + \hat{B} + \hat{C} = \pi + \Theta$

Consider the geometric surface corresponding to a sphere *S*, described in the Example 14A. In the present example we are going to check relation 15.9d, for the case of a geodesic triangle of S. First we show that the meridians circles and the equator are geodesic curves of *S*. Next we consider a triangle *ABC* on *S* determined by two meridians and the equator. Let *A* be the pole of the sphere and *B*, *C* the intersecting points of two meridians with the equator (figure 15.2). It is evident that each of the angles $\hat{B}$ and $\hat{C}$ of the triangle equals $\pi/2$ rad. Hence, If $\hat{A} = \Theta$ ($\hat{A}$ is the angle determined by the tangents of the two meridians at the pole *A*), we anticipate that the sum of the angles of the triangle *ABC* equals $\pi + \Theta$ rad: $\hat{A} + \hat{B} + \hat{C} = \pi + \Theta$

We are going to confirm this anticipation, by applying the general equation 15.9d in this particular case.

A) Geodesic curves on the sphere *S*

Let $c : c(t) = \left(c^1(t), c^2(t)\right)$ be a curve of the parameter-space, corresponding to a geodesic *C* of the sphere *S*. The parameter *t* is arbitrary. According to paragraph 13, if the image *C* of *c*

is to be a geodesic of S, $c^1(t)$, $c^2(t)$ should be solutions of the differential equations resulting by the consequent equations:

$\Delta c = \dot{c}(t)\Delta t\,,\ \Delta t \to 0$

$D_{\Delta c}\left(e_\nu(c(t))\dot{c}^\nu(t)\right) = 0$

$D_{\Delta c}\left(e_\nu(c(t))\dot{c}^\nu(t) + e_\nu(c(t))\ddot{c}^\nu(t)\Delta t = 0\right.$[4]

$\ddot{c}^\nu + \bar{\Gamma}^\nu_{\mu\kappa}\dot{c}^\mu\dot{c}^\kappa = 0$

For the case of the sphere, the symbols $\bar{\Gamma}^\nu_{\mu\kappa}$ have been determined in 12A.3; we find that $c^1(t)$, $c^2(t)$ are solutions of the equations:

$$\ddot{c}^1 - \frac{2c^2}{b^2-(c^2)^2}\dot{c}^1\dot{c}^2 = 0 \qquad \text{(15B.1a)}$$

$$\ddot{c}^2 + \frac{c^2}{b^2}\left(b^2-(c^2)^2\right)\left(\dot{c}^1\right)^2 + \frac{c^2}{b^2-(c^2)^2}\left(\dot{c}^2\right)^2 = 0 \qquad \text{(15B.1b)}$$

From the explicit form of 15B.1a and b, we can easily verify that the following curves are solutions:

a) The curve defined by the equations:

$$\begin{aligned}\ddot{c}^1 &= 0\\ c^2 &= 0\end{aligned} \qquad \text{(15B.2)}$$

This curve of the parameters' space corresponds to the equator of the sphere; i.e. the equator is a geodesic curve of the sphere S.

b) Let us try to test if a curve with analytic expression: $c^1 = \Theta = \text{constant}$ is a geodesic of S:

b1) equation 15B.1a is satisfied,

b2) the second equation (15B.1b) takes the form:

$$\ddot{c}^2 + \frac{c^2}{b^2-(c^2)^2}\left(\dot{c}^2\right)^2 = 0 \qquad \text{(15B.3)}$$

Let us identify the abstract parameter t with the parameter s, which represent the length of the curve from any of its fixed point P_0. We set $c^2(t)=f(s)$ and we obtain the subsequent equations:

$f' \underset{def}{=} \dfrac{df}{ds}$

$f''^2 + \dfrac{c^2}{b^2-f^2}f'^2 = 0$

$\dot{c}^2(t)\Delta t = f'(s)\Delta s$

$\left(\Delta s\right)^2 = g_{\mu\nu}\dot{c}^\mu\dot{c}^\nu\left(\Delta t\right)^2 = \dfrac{b^2}{b^2-(c^2)^2}\left(\dot{c}^2\right)\left(\Delta t\right)^2 = \dfrac{b^2}{b^2-f^2}\left(f'\Delta s\right)^2$

$f' = \dfrac{1}{b}\sqrt{b^2-f^2}$

$f'' = -\dfrac{1}{b}\left(b^2-f^2\right)^{-1/2}ff'$

We imply that f is a solution of the differential equation:

---

[4] According to 11.20, for any differentiable real function $F(\dot{c})$ we write: $D_{\Delta c}F(\dot{c}) = \dfrac{\partial F(\dot{c})}{\partial \dot{c}^\nu}\ddot{c}^\nu\Delta t$

For $F(\dot{c}) = \dot{c}^\lambda$ we obtain: $D_{\Delta c}\dot{c}^\lambda = \delta^\lambda_\nu\ddot{c}^\nu\Delta t = \ddot{c}^\lambda\Delta t$

$$f\left(b\left(b^2-f^2\right)^{-1/2}f'-1\right)=0$$

We reject the option f=0 and we reduce the last equation to the next:

$$\frac{df}{ds}=\frac{1}{b}\sqrt{b^2-f^2} \tag{15B.4}$$

The solution of 15B.4 is given by the analytic expression:

$$f(s)=b\cos\left(\cos^{-1}\left(\frac{f(0)}{b}\right)-\frac{1}{b}s\right) \tag{15B.5a}$$

We choose $f(0)=-b$ and 15B.5a takes the form:

$$f(s)=-b\cos\left(\frac{1}{b}s\right) \tag{15B.5b}$$

We infer that the analytic expression of the curve c of the parameters' space corresponding to a geodesic of S is:

$$c(s)=\left(c^1(s),c^2(s)\right)=\left(\Theta,-b\cos\left(\frac{1}{b}s\right)\right) \tag{15B.6}$$

These geodesics are identical to the semi-meridians of the sphere, specified by the values of the constant angle Θ (figure 15.2).

The parameter s is the length along a semi-meridian; its maximum value is obtained for:

$c^2(s_{max})=b$

$\cos\left(\frac{1}{b}s_{max}\right)=-1$

Hence:

$s_{max}=\pi b$

For a point on the equator, we have:

$c^2(s)=0$

$\cos\left(\frac{1}{b}s_1\right)=0$

$s_1=\frac{\pi b}{2}$

It is clear that all the previous results are been anticipated.

B) Calculation of the sum of the angles of the geodesic triangle ABC

According to the Gauss-Bonnet theorem, the sum $\hat{A}+\hat{B}+\hat{C}$ of the triangle ABC illustrated in figure 15.2 is calculated by applying the relationship 15.9c. For the case of our application, the consequent relations are true:

$$\hat{A}+\hat{B}+\hat{C}=\pi+\iint\limits_{(P_1P_2P_3)}K(u)\,\mathrm{area}\left(\Delta\Pi_u\right) \tag{15B.7}$$

$K(u)=\frac{1}{b^2}$

$\det g(u)=b^2$

$\mathrm{area}\left(\Delta\Pi_u\right)=\sqrt{\det g(u)}\,\omega^1\wedge\omega^2\left(\Delta_{(1)}U,\Delta_{(2)}U\right)$

$\Delta_{(1)}U=e_1(u)\Delta u^1$, $\Delta_{(2)}U=e_2(u)\Delta u^2$

Hence:

$\mathrm{area}\left(\Delta\Pi_u\right)=b\Delta u^1\Delta u^2$

The parameter u^1 runs the values from 0 to Θ and the length s runs from $\frac{\pi b}{2}$ to πb

On the other hand:

$$u^2 = -b\cos\left(\frac{1}{b}s\right),\ \Delta u^2 = \sin\left(\frac{1}{b}s\right)\Delta s$$

We are now able to calculate the integral in 15B.7:

$$\hat{A}+\hat{B}+\hat{C} = \pi + \iint\limits_{(P_1P_2P_3)} \frac{1}{b^2} b\Delta u^1 \Delta u^2$$

$$\hat{A}+\hat{B}+\hat{C} = \pi + \frac{1}{b}\int\limits_{u^1=0}^{\Theta} \Delta u^1 \int\limits_{s=\pi b/2}^{\pi b} \Delta s \sin\left(\frac{1}{b}s\right)$$

$$\hat{A}+\hat{B}+\hat{C} = \pi + \frac{\Theta}{b}\left[-b\cos\left(\frac{1}{b}s\right)\right]_{s=\pi b/2}^{s=\pi b} = \pi + \Theta(1+0) = \pi + \Theta$$

This final result confirms the anticipation we made in the beginning of the present Example 15B.

APPENDICES

Appendix 1

Composition of one-parameter groups of coordinate-transformations that leave invariant a given real function defined on the tangent spaces of a linear space or a geometric surface
The case of the Euclidean coordinate-transformations in the Euclidean plane and the Lorentz transformations in the Minkowski plane

Key concepts: The group of diffeomorphic coordinate-transformations - One-parameter group of coordinate transformation - The group of matrices on the tangent spaces of a two-dimensional Euclidean or pseudo-Euclidean space corresponding to the one-parameter group of coordinate-transformations - Generators of the one parameter groups of transformations - Infinitesimal coordinate-transformations - Composition of the one parameter group of coordinate-transformations by means of the generators of the group - Evaluation of the group-generators from the invariant functions - Example 1: The Euclidean transformations in the 2-dimensional Euclidean space - Example 2: The Lorentz transformations in the 2-dimensional Minkowski space

The group of diffeomorphic coordinate-transformations

Consider a real function of the form:

$G(x;\Delta x)$, $x \in \boldsymbol{R}_0^2$, $\Delta x \in T_x\boldsymbol{R}_0^2$

The points of the Euclidean plane $\boldsymbol{R}_0^2$ and the vectors of its tangent spaces $T_x\boldsymbol{R}_0^2$ are represented in Cartesian coordinates:

$x = (x^1, x^2)$, $\Delta x = (\Delta x^1, \Delta x^2) = \boldsymbol{x}_j \Delta x^j$

The "natural" basis at every tangent space is consisted by the vectors:

$\boldsymbol{x}_1 = (1,0)$, $\boldsymbol{x}_2 = (0,1)$

We are going to develop a procedure to compose the one-parameter groups of coordinate-transformations that leave invariant functions of the form $G(x;\Delta x)$ which are continues with respect to all of their arguments and have derivatives at least up to the second order.

Let us assume the diffeomorphic coordinate-transformation:

$$\bar{x}^j = \bar{x}^j(x^1, x^2) \tag{A1.1}$$

In paragraph 5, we have seen that the diffeomorphic coordinate-transformations are invertible and they have derivatives of at least up to the second order. We symbolize $\mathrm{Diff}(\boldsymbol{R}_0^2)$ the set of the diffeomorphic coordinate-transformations of the two dimensional Euclidean space. The set $\mathrm{Diff}(\boldsymbol{R}_0^2)$ equipped with the operation of transformation-composition acquires the structure of a group [(3)].

The group of the Jacobian-matrices of the coordinate-transformations

Under the transformation 1 the coordinates of the vectors $\Delta x \in T_x\boldsymbol{R}_0^2$ transform according to the relation:

$$\Delta\bar{x}^k = R_j^k \Delta x^j \tag{A1.2a}$$

$$R_j^k(x) \underset{def}{=} \frac{\partial \bar{x}^k(x)}{\partial x^j} = \partial_j \bar{x}^k(x) \tag{A1.2b}$$

We use the symbolism: $\partial_k = \frac{\partial}{\partial x^k}$, $\bar{\partial}_k = \frac{\partial}{\partial \bar{x}^k}$,...

The matrix $R(x) \underset{def}{=} \left[R_j^k(x)\right]$ is the Jacobian matrix of the transformation given by A1.1.

The coordinate-transformations $\bar{x}^j = \bar{x}^j(x^1, x^2)$ are diffeomorphisms (see paragraph 5); hence the Jacobian matrix R of each of them is invertible. The Jacobian matrix of the inverse transformation equals the inverse of the matrix R:

$$\bar{R}(\bar{x}) = \left[\frac{\partial x^k(\bar{x})}{\partial \bar{x}^j}\right] = \left[\bar{\partial}_j x^k(\bar{x})\right]$$

$$x^j = x^j\left(\bar{x}^j(x^1, x^2)\right)$$

$$\frac{\partial x^j}{\partial x^k} = \frac{\partial x^j}{\partial \bar{x}^m}\frac{\partial \bar{x}^m}{\partial x^k}$$

$$\delta_k^j = \bar{R}_m^j R_k^m$$

Furthermore, consider the composed transformation:

$$x^j = x^j\left(\bar{x}^j\left(\bar{\bar{x}}\right)\right)$$

Its matrix is determined by the relationship:

$$\left[\bar{\bar{R}}_k^j\right] = \left[\frac{\partial x^j}{\partial \bar{\bar{x}}^k}\right]$$

We symbolize:

$$\left[\bar{R}_j^k\right] = \left[\frac{\partial x^k(\bar{x})}{\partial \bar{x}^j}\right], \left[Q_j^k\right] = \left[\frac{\partial \bar{x}^k\left(\bar{\bar{x}}\right)}{\partial \bar{\bar{x}}^j}\right]$$

We obtain the identity:

$$\bar{\bar{R}}_k^j = \bar{R}_m^j Q_k^m$$

By using the previous identities, we can easily verify [3], [8] that in every tangent space of the Euclidean plane the set of the matrices $\left[R_j^k(x)\right] = \left[\partial_j \bar{x}^k(x)\right]$ equipped with the operation of the matrix-multiplication, acquires the structure of a group.

One-parameter group of coordinate-transformation - Generators

Consider the one-parameter group of the coordinate-transformations

$$\bar{x}^j = h^j(x;\varphi),\ \varphi \in \boldsymbol{R} \qquad \text{(A1.3)}$$

The parameter φ has been chosen so that if $\bar{\bar{x}}^j = h^j\left(\bar{x};\bar{\varphi}\right)$ then the composite transformation $\bar{\bar{x}}^j = h^j\left(h^j(x;\varphi);\bar{\varphi}\right)$ is given by the rule [8]:

$$\bar{\bar{x}}^j = h^j\left(h^j(x;\varphi);\bar{\varphi}\right) = h^j\left(x;\varphi+\bar{\varphi}\right)$$

Furthermore, the following identities hold:

$$x^j = h^j(x;0)$$

$$x^j = h^j\left(\bar{x};-\varphi\right)$$

The one-parameter transformations A1.3 form an Abelian group with operation the composition. The Jacobian matrix of h is:

$$\left[R_j^k(x;\varphi)\right] = \left[\partial_j h^k(x;\varphi)\right], \left[R_j^k(x;0)\right] = \left[\delta_j^k\right]$$

Consider an infinitesimal variation $\delta\varphi \to 0$ of the parameter. This variation of φ causes an infinitesimal variation $\delta\bar{x}$ of the $\bar{x}$-coordinates, where:

$\bar{x} = h(x;\varphi) = \left(h^1(x;\varphi), h^2(x;\varphi), h^3(x;\varphi)\right)$

The infinitesimal variation of the $\bar{x}$-coordinate system is determined by the equations:

$$\bar{x}^j + \delta\bar{x}^j = h^j(x;\varphi+\delta\varphi) = h^j\left(h^j(x;\varphi);\delta\varphi\right) = h^j(\bar{x};\delta\varphi) \tag{A1.4a}$$

From A1.4a, by holding terms up to first order in the Taylor expansions, we infer that:

$$\delta\bar{x}^j = \frac{\partial h^j(x;\varphi)}{\partial\varphi}\delta\varphi = \left.\frac{\partial h^j(\bar{x};\psi)}{\partial\psi}\right|_{\psi=0}\delta\varphi \tag{A1.4b}$$

The quantities $\eta^j(h) = \left.\frac{\partial h^j(h;\psi)}{\partial\psi}\right|_{\psi=0}$ are called "**generators**" of the group of the transformations A1.3: if we know the functions $\eta^j(x)$ we are able to compose the transformation group by solving the differential equations:

$$\frac{\partial h^j(x;\varphi)}{\partial\varphi} = \eta^j\left(h(x;\varphi)\right),\ h^j(x;0) = x^j \tag{A1.5}$$

Proposition A1.1

The following identity holds:

$$\eta^j\left(h(x;\varphi)\right) = \partial_k h^j(x;\varphi)\eta^k(x) \tag{A1.6}$$

Steps to the proof

$$\eta^j(x) = \left.\frac{\partial h^j(x;\bar{\varphi})}{\partial\bar{\varphi}}\right|_{\bar{\varphi}=0}$$

$$\eta^j\left(h(x;\varphi)\right) = \left.\frac{\partial}{\partial\bar{\varphi}}h^j\left(h(x;\varphi);\bar{\varphi}\right)\right|_{\bar{\varphi}=0} = \left.\frac{\partial}{\partial\bar{\varphi}}h^j(x;\varphi+\bar{\varphi})\right|_{\bar{\varphi}=0} =$$

$$= \left.\frac{\partial}{\partial\bar{\varphi}}h^j\left(h(x;\bar{\varphi});\varphi\right)\right|_{\bar{\varphi}=0} = \left.\frac{\partial}{\partial\bar{\varphi}}h^j\left(x+\bar{\varphi}\left.\frac{\partial h(x;\psi)}{\partial\psi}\right|_{\psi=0};\varphi\right)\right|_{\bar{\varphi}=0} =$$

$$= \left.\frac{\partial}{\partial\bar{\varphi}}\left(h^j(x;\varphi)+\partial_k h^j(x;\varphi)\left.\frac{\partial h^k(x;\psi)}{\partial\psi}\right|_{\psi=0}\bar{\varphi}\right)\right|_{\bar{\varphi}=0} =$$

$$= \partial_k h^j(x;\varphi)\eta^k(x)$$

•

Infinitesimal coordinate-transformations

Relation A1.4a can be written in the form:

$$\begin{aligned}\bar{x}^j + \delta\bar{x}^j &= h^j(x;\varphi+\delta\varphi) = h^j\left(h(x;\delta\varphi);\varphi\right) = \\ &= h^j\left(x+\left.\frac{\partial h^j(x;\bar{\varphi})}{\partial\bar{\varphi}}\right|_{\bar{\varphi}=0}\delta\varphi;\varphi\right) = h^j\left(x+\eta(x)\delta\varphi;\varphi\right)\end{aligned} \tag{A1.7a}$$

Hence:

$$\bar{x}^j + \delta\bar{x}^j = h^j(\bar{x};\delta\varphi) = \bar{x}^j + \bar{\eta}^j(\bar{x})\delta\varphi \tag{A1.7b}$$

The coordinates $\bar{x}^j$ are mapped to the nearby coordinates: $\bar{x}^j + \delta\bar{x}^j$

The coordinate-transformation expressed by A1.7b is called "**infinitesimal transformation**" of the group.

Evaluation of the group-generators from the invariant functions

Assume that the coordinate-transformation $\bar{x}^j = h^j(x;\varphi)$ leaves invariant the real functions:

$G^{(a)}(x;\Delta x)$, $a = 1,2...$, $x \in \boldsymbol{R}_0^2$, $\Delta x \in T_x\boldsymbol{R}_0^2$

Then, for every value of the parameter φ we have:

$$G^{(a)}(x;\Delta x) = G^{(a)}\left(h(x;\varphi);\partial_k h(x;\varphi)\Delta x^k\right) \qquad \text{(A1.8a)}$$

For the infinitesimal transformation $\bar{x}^j + \delta\bar{x}^j = h^j(\bar{x};\delta\varphi)$ A1.8a takes the form

$$G^{(a)}(\bar{x};\Delta\bar{x}) = G^{(a)}\left(h(\bar{x};\delta\varphi);\bar{\partial}_k h(\bar{x};\delta\varphi)\Delta\bar{x}^k\right) \qquad \text{(A1.8b)}$$

By using A1.b, we calculate the Jacobian matrix of the infinitesimal transformation:

$\bar{x}^j + \delta\bar{x}^j = h^j(\bar{x};\delta\varphi)$

In the $\bar{x}$-coordinate system consider the curve:

$\bar{c} : \bar{x}^j(\sigma) = \bar{c}^j(\sigma)$, $\bar{c}^j(0) = \bar{x}^j$

The tangent vector of $\bar{c}$ at $\bar{x} = (\bar{x}^1, \bar{x}^2)$ is a vector of the tangent space $T_{\bar{x}}\boldsymbol{R}_0^2$ with coordinates:

$$\dot{\bar{x}}^j = \left.\frac{d\bar{x}^j(\sigma)}{d\sigma}\right|_{\sigma=0}$$

In the $(\bar{x} + \delta\bar{x})$-coordinate system the previous curve has the analytic form:

$\tilde{x}^j(\sigma) = h^j(\bar{x}(\sigma);\delta\varphi)$

Hence, the coordinates of the same tangent vector in the $(\bar{x} + \delta\bar{x})$-coordinate system are calculated by the subsequent equations (we keep terms up to the first order in the corresponding Taylor expansions):

$$\dot{\tilde{x}}^j(\sigma) = \left.\frac{d}{d\sigma}h^j(\bar{x}(\sigma);\delta\varphi)\right|_{\sigma=0} =$$

$$= \bar{\partial}_k h^j(\bar{x};\delta\varphi)\left.\frac{d\bar{x}^k(\sigma)}{d\sigma}\right|_{\sigma=0} = \bar{\partial}_k\left(\bar{x}^j + \eta^j(\bar{x})\delta\varphi\right)\dot{\bar{x}}^k = \left(\delta_k^j + \bar{\partial}_k\eta^j(\bar{x})\delta\varphi\right)\dot{\bar{x}}^k =$$

$$= \dot{\bar{x}}^j + \delta\varphi\bar{\partial}_k\eta^j(\bar{x})\dot{\bar{x}}^k$$

We conclude that the Jacobian matrix of the infinitesimal transformation $\bar{x}^j + \delta\bar{x}^j = h^j(\bar{x};\delta\varphi)$ is given by the matrix: $\left[\delta_k^j + \delta\varphi\bar{\partial}_k\eta^j(\bar{x})\right]$

Now, we come back to the equation A1.8b; by expanding the function at its right hand side in a Taylor series with respect to the infinitesimal quantity $\delta\varphi$ we obtain:

$$G^{(a)}(\bar{x};\Delta\bar{x}) = G^{(a)}\left(\bar{x} + \eta(\bar{x})\delta\varphi;\Delta\bar{x} + \delta\varphi\partial_k\bar{\eta}(\bar{x})\Delta\bar{x}^k\right)$$

$$\frac{\partial G^{(a)}(\bar{x};\Delta\bar{x})}{\partial\bar{x}^j}\eta^j(\bar{x}) + \frac{\partial G^{(a)}(\bar{x};\Delta\bar{x})}{\partial\Delta\bar{x}^j}\bar{\partial}_k\eta^j(\bar{x})\Delta\bar{x}^k = 0 \qquad \text{(A1.9)}$$

Where: $\Delta\bar{x} = \dot{\bar{x}}(\sigma)\Delta\sigma$

The equations expressed by A1.9 are the key point which may lead us to the determination of the analytic form of the generators $\eta^j(\bar{x})$ of the coordinate-transformations. Then, by solving the differential equations A1.5, we could construct the one parameter groups of transformations

which leave invariant the real functions $G^{(a)}$. We are going to apply this general procedure for the cases of the Euclidean and Lorentz transformations.

Example A1: The Euclidean transformations in the 2-dimensional Euclidean space

Let us apply A1.9 and A1.5 to derive the Euclidean transformations in the two-dimensional Euclidean space. By definition, the Euclidean transformations are the coordinate-transformations that leave invariant the analytic expression of the Euclidean inner product given in Cartesian coordinates. I.e. the form of the real function that is to be invariant under the two-dimensional Euclidean transformations is the following:

$$G\left(\bar{x};\Delta\bar{x}\right)=\left(\Delta\bar{x}^1\right)^2+\left(\Delta\bar{x}^2\right)^2 \tag{A1.10}$$

The coordinates $\bar{x}$ are assumed to be Cartesian.

We apply A1.9 for the analytic form of G given by A1.10 and we obtain:

$$\partial_k\bar{\eta}^j\left(\bar{x}\right)\Delta\bar{x}^j\Delta\bar{x}^k=0 \tag{A1.11}$$

$$\left(\Delta\bar{x}^1\right)^2\bar{\partial}_1\eta^1\left(\bar{x}\right)+\left(\Delta\bar{x}^2\right)^2\bar{\partial}_2\eta^2\left(\bar{x}\right)+\Delta\bar{x}^1\Delta\bar{x}^2\left(\bar{\partial}_1\eta^2\left(\bar{x}\right)+\bar{\partial}_2\eta^1\left(\bar{x}\right)\right)=0 \tag{A1.12}$$

Equation A1.12 holds for every value of $\Delta\bar{x}^1$, $\Delta\bar{x}^2$ hence, the following relationships are valid:

$$\bar{\partial}_1\eta^1\left(\bar{x}\right)=0,\ \bar{\partial}_2\eta^2\left(\bar{x}\right)=0,\ \bar{\partial}_1\eta^2\left(\bar{x}\right)+\bar{\partial}_2\eta^1\left(\bar{x}\right)=0 \tag{A1.12a}$$

$$\eta^1\left(\bar{x}\right)=\eta^1\left(\bar{x}^2\right),\ \eta^2\left(\bar{x}\right)=\eta^2\left(\bar{x}^1\right),\ \bar{\partial}_1\eta^2\left(\bar{x}^1\right)=c\ \ \bar{\partial}_2\eta^1\left(\bar{x}^2\right)=-c \tag{A1.12b}$$

The real constant c is arbitrary; nevertheless we can choose the system of units so that c=1. Then, the matrix $\left[\partial_k\eta^j\left(x\right)\right]$ takes the form:

$$\left[\partial_k\eta^j\left(x\right)\right]=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \tag{A1.13a}$$

By solving equations A1.12a and b we derive the generators of the group:

$$\eta^1\left(\bar{x}\right)=\eta^1\left(\bar{x}^2\right)=-\bar{x}^2+a^1,\ \eta^2\left(\bar{x}\right)=\eta^2\left(\bar{x}^1\right)=\bar{x}^1+a^2 \tag{A1.13b}$$

The values of the constants a^1 and a^2 are arbitrary.

According to the previous relationships, the analytic expressions of the generators of the group have been specified; the last step is to apply equation A1.5 for the case of the generators given by A1.13b, and solve it. By its solution we shall determine the one-parameter group of the transformations that leave invariant the analytic expression of the real function defined by A1.10:

$$\frac{\partial h^1\left(x;\varphi\right)}{\partial\varphi}=-h^2\left(x;\varphi\right)+a^1 \tag{A1.14a}$$

$$\frac{\partial h^2\left(x;\varphi\right)}{\partial\varphi}=h^1\left(x;\varphi\right)+a^2 \tag{A1.14b}$$

The initial conditions are:

$h^1\left(x;0\right)=x^1$, $h^2\left(x;0\right)=x^2$

The solution of the system A1.14a and b that satisfy the imposed initial conditions is given by the functions:

$$h^1\left(x;\varphi\right)=\left(x^1+\mathrm{a}^2\right)\cos\varphi+\left(-x^2+a^1\right)\sin\varphi-a^2 \tag{A1.15a}$$

$$h^2\left(x;\varphi\right)=\left(x^1+\mathrm{a}^2\right)\sin\varphi-\left(-x^2+a^1\right)\cos\varphi+a^1 \tag{A1.15b}$$

For a^1=a^2=0 we obtain the well-known group of rotations in the Euclidean plane:

$$\begin{aligned}h^1\left(x;\varphi\right)&=x^1\cos\varphi-x^2\sin\varphi\\ h^2\left(x;\varphi\right)&=x^1\sin\varphi+x^2\cos\varphi\end{aligned} \tag{A1.16a}$$

The Jacobian matrix of the transformation A1.16a is independent of the position x of the tangent space:

$$\left[R_k^j(x;\varphi)\right]=\left[\partial_k h^j(x;\varphi)\right]=\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix} \tag{A1.16b}$$

Example A2: The Lorentz transformations in the 2-dimensional Minkowski space

The Lorentz transformations $\bar{x}^j=h^j(x;\varphi)$ leave invariant the analytic expression of the Minkowski inner product given in Cartesian coordinates, at any tangent plane of the Minkowski plane. I.e. the form of the real function that is to be invariant under the two-dimensional Lorentz transformations is the following:

$$G(\bar{x};\Delta\bar{x})=\left(\Delta\bar{x}^1\right)^2-\left(\Delta\bar{x}^2\right)^2 \tag{A1.17}$$

We apply the procedure we followed in Example A1; its implementation is specified by the following steps:

a) Derivation of the Jacobian matrices of the Lorentz transformations from the generators of the group

The matrix-elements $R_k^j(x;\varphi)$ of the Lorentz transformations $\bar{x}^j=h^j(x;\varphi)$ satisfy the following relations:

$$R_k^j(x;\varphi+\delta\varphi)=\frac{\partial h^j(x;\varphi+\delta\varphi)}{\partial x^k}=\frac{\partial h^j(\bar{x};\delta\varphi)}{\partial x^k},\ \delta\varphi\to 0$$

$$R_k^j(x;\varphi)+\delta\varphi\frac{\partial R_k^j(x;\varphi)}{\partial\varphi}=\frac{\partial}{\partial x^k}\left(h^j(\bar{x};0)+\delta\varphi\frac{\partial h^j(\bar{x};\bar{\varphi})}{\partial\bar{\varphi}}\bigg|_{\bar{\varphi}=0}\right)$$

$$R_k^j(x;\varphi)+\delta\varphi\frac{\partial R_k^j(x;\varphi)}{\partial\varphi}=\frac{\partial}{\partial x^k}\left(h^j(x;\varphi)+\delta\varphi\eta^j(\bar{x})\right)$$

$$\frac{\partial R_k^j(x;\varphi)}{\partial\varphi}=\frac{\partial\eta^j(\bar{x})}{\partial x^k}=\frac{\partial\eta^j(\bar{x})}{\partial\bar{x}^m}\frac{\partial h^m(x;\varphi)}{\partial x^k}$$

$$\frac{\partial R_k^j(x;\varphi)}{\partial\varphi}=\bar{\partial}_m\eta^j(\bar{x})R_k^m(x;\varphi) \tag{A1.18}$$

The matrix $\left[\bar{R}_k^j(\bar{x};-\varphi)\right]=\left[\bar{\partial}_k h^m(\bar{x};-\varphi)\right]$ is the inverse of the matrix: $\left[R_k^j(x;\varphi)\right]=\left[\partial_k h^j(x;\varphi)\right]$

Indeed, we have:

$$\bar{R}_k^j(\bar{x};-\varphi)R_m^k(x;\varphi)=\frac{\partial h^j(\bar{x};-\varphi)}{\partial\bar{x}^k}\frac{\partial h^k(x;\varphi)}{\partial x^m}=\frac{\partial h^j(\bar{x};-\varphi)}{\partial x^m}=$$

$$=\frac{\partial h^j\left(h(x;\varphi);-\varphi\right)}{\partial x^m}=\frac{\partial h^j(x;0)}{\partial x^m}=\frac{\partial x^j}{\partial x^m}=\delta_m^j$$

Multiply both sides of A1.18 by $\bar{R}_p^k(\bar{x};-\varphi)$ and add with respect to the index k. We obtain the consequent equations:

$$\frac{\partial R_k^j(x;\varphi)}{\partial\varphi}\bar{R}_p^k(\bar{x};-\varphi)=\bar{\partial}_m\eta^j(\bar{x})R_k^m(x;\varphi)\bar{R}_p^k(\bar{x};-\varphi)$$

$$\frac{\partial}{\partial\varphi}\left(R_k^j(x;\varphi)\bar{R}_p^k(\bar{x};-\varphi)\right)-R_k^j(x;\varphi)\frac{\partial}{\partial\varphi}\bar{R}_p^k(\bar{x};-\varphi)=\bar{\partial}_p\eta^j(\bar{x})$$

$$-R_k^j(x;\varphi)\frac{\partial}{\partial\varphi}\bar{R}_p^k(\bar{x};-\varphi)=\bar{\partial}_p\eta^j(\bar{x})$$

Multiply both sides of the last equation with $\bar{R}_j^q(\bar{x};-\varphi)$ and add with respect to *j*:

$$-\frac{\partial}{\partial\varphi}\bar{R}_p^k(\bar{x};-\varphi)=\bar{R}_j^q(\bar{x};-\varphi)\bar{\partial}_p\eta^j(\bar{x})$$

By changing the parameter $\varphi\to-\varphi$ we finally result the equation:

$$\frac{\partial}{\partial\varphi}\bar{R}_p^q(\bar{x};\varphi)=\bar{R}_j^q(\bar{x};\varphi)\bar{\partial}_p\eta^j(\bar{x}) \qquad \text{(A1.19)}$$

We conclude that the matrices $\left[\bar{R}_p^q(\bar{x};\varphi)\right]$ of the transformations of the group are to be obtained by solving the system of the differential equations A1.19, with initial condition:

$\left[\bar{R}_p^q(\bar{x};0)\right]=I$

The solution of the equations A1.19 is possible to be accomplished if we know the analytic expressions of the quantities $\bar{\partial}_p\eta^j(\bar{x})$ which are to be derived in the next section (b) of the present example.

b) Derivation of the quantities $\bar{\partial}_p\eta^j(\bar{x})$ from the invariant function *G*

We apply equation A1.9 for the case of the invariant function given by A1.17. By following the steps of the Example 1 we obtain the consequent form for the matrix $\left[\bar{\partial}_p\eta^j(\bar{x})\right]$

$$\left[\bar{\partial}_p\eta^j(\bar{x})\right]=\begin{pmatrix}0&1\\1&0\end{pmatrix}\underset{def}{=}\sigma \qquad \text{(A1.20)}$$

According to A1.20, equation A1.19 is written:

$$\frac{\partial\bar{R}(\varphi)}{\partial\varphi}=\bar{R}(\varphi)\sigma,\ \bar{R}(0)=I \qquad \text{(A1.21)}$$

c) Derivation of the matrix $\bar{R}(\varphi)$

The last task is to solve the matrix-differential equation A1.21. This is achieved by expanding the matrix $\bar{R}(\varphi)$ in a Taylor series with respect to the parameter φ [9].

$$\bar{R}(\varphi)=I+\varphi\left.\frac{\partial\bar{R}}{\partial\varphi}\right|_{\varphi=0}+\frac{\varphi^2}{2!}\left.\frac{\partial^2\bar{R}}{\partial\varphi^2}\right|_{\varphi=0}+\frac{\varphi^3}{3!}\left.\frac{\partial^3\bar{R}}{\partial\varphi^3}\right|_{\varphi=0}+\ldots$$

By using A1.21, we result that:

$$\bar{R}(\varphi)=I+\varphi\sigma+\frac{\varphi^2}{2!}\sigma^2+\frac{\varphi^3}{3!}\sigma^3+\ldots$$

$$\bar{R}(\varphi)=I\left(1+\frac{\varphi^2}{2!}+\frac{\varphi^4}{4!}+\ldots\right)+\sigma\left(\varphi+\frac{\varphi^3}{3!}+\frac{\varphi^5}{5!}+\ldots\right)$$

The property $\sigma^2=I$ of the matrix σ has been used.

$$\bar{R}(\varphi)=I\cosh(\varphi)+\sigma\sinh(\varphi) \qquad \text{(A1.22a)}$$

$$\bar{R}(\varphi)=\begin{pmatrix}\cosh(\varphi)&\sinh(\varphi)\\\sinh(\varphi)&\cosh(\varphi)\end{pmatrix} \qquad \text{(A1.22b)}$$

The coordinates-transformation that leaves the analytic form of the Minkowski metric tensor invariant is accomplished by solving equation:

$$\frac{\partial x^j}{\partial\bar{x}^k}=\bar{R}_k^j(-\varphi)$$

We obtain the linear transformations:

$$\bar{x}^1 = x^1 \cosh(\varphi) + x^2 \sinh(\varphi)$$
$$\bar{x}^2 = x^1 \sinh(\varphi) + x^2 \cosh(\varphi) \qquad \text{(A1.23)}$$

We can see that under the transformations A1.23, the straight line $x^2 = 0$ is mapped to the straight line:

$$\frac{\bar{x}^2}{\bar{x}^1} = \tanh(\varphi)$$

We define the slope $\tanh(\varphi)$ of this line as a new parameter u and we replace φ in A1.23, according to the relationship:

$$u = c\tanh(\varphi)$$

The constant c is depended on the choice of the System of Units; the analytic form of A1.23 changes to the well-known Lorentz transformations:

$$\bar{x}^1 = \frac{1}{\sqrt{1 - u^2/c^2}}\left(x^1 + x^2 \frac{u}{c}\right)$$

$$\bar{x}^2 = \frac{1}{\sqrt{1 - u^2/c^2}}\left(x^1 \frac{u}{c} + x^2\right)$$

Appendix 2

Infinitesimal orthogonal coordinate-transformation in the 3-dimensional Euclidean space

Let us consider the group of the orthogonal coordinate-transformations with Jacobian matrices $R = \left[R_k^j\right]$ in a 3-dimensional Euclidean space.

For the case of an infinitesimal transformation, R is infinitely close to the identity matrix:

$$R = I + \delta\varphi\Omega,\ I = \left[\delta_k^j\right],\ \delta\varphi \to 0 \qquad \text{(A2.1)}$$

In the present appendix, the matrix Ω is to be determined.

Given that the orthogonal transformations leave invariant the analytic expression of the Euclidean inner product of the space, expressed in Cartesian coordinates (see paragraph 5), their Jacobian matrices satisfy the condition:

$$g = R^T g R$$

In Cartesian coordinates, the matrix g of the metric tensor is:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence, from A2.1, we imply that:

$$\left(I + \delta\varphi\Omega\right)^T \left(I + \delta\varphi\Omega\right) = I \qquad \text{(A2.2)}$$

$$\Omega + \Omega^T = 0 \qquad \text{(A2.3)}$$

By A2.3 we infer that the matrix Ω is antisymmetric; it has the form:

$$\Omega = \begin{pmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{pmatrix} \qquad \text{(A2.5)}$$

The signs + and - have been chosen so that the coordinates of the vector $\Omega\,\Delta x$ be identical with the coordinates of the exterior product $\omega \otimes \Delta x$ i.e.:

$$\Omega\begin{pmatrix}\Delta x^1\\ \Delta x^2\\ \Delta x^3\end{pmatrix}=\begin{pmatrix}-\omega^3\Delta x^2+\omega^2\Delta x^3\\ \omega^3\Delta x^1-\omega^1\Delta x^3\\ -\omega^2\Delta x^1+\omega^1\Delta x^2\end{pmatrix}$$

$$\omega\otimes\Delta x=\left(\boldsymbol{x}_j\omega^j\right)\otimes\left(\boldsymbol{x}_k\Delta x^k\right)=$$

$$=\boldsymbol{x}_1\left(\omega^2\Delta x^3-\omega^3\Delta x^2\right)+\boldsymbol{x}_2\left(\omega^3\Delta x^1-\omega^1\Delta x^3\right)+\boldsymbol{x}_3\left(\omega^1\Delta x^2-\omega^2\Delta x^1\right)$$

As usually, the vectors: $\boldsymbol{x}_j$ j=1,2,3 consist the "natural" basis of the Euclidean space:

$\Delta x=\boldsymbol{x}_k\Delta x^k$, $\omega=\boldsymbol{x}_j\omega^j$

$$\boldsymbol{x}_j\otimes\boldsymbol{x}_k=\sum_{n=1}^{3}\varepsilon_{jkn}\boldsymbol{x}_n$$

The quantity ε_{jkn} equals to zero if any two of the values of *j, k, n* are equal. If $j\neq k\neq n$ then ε_{jkn} equals to the parity of the permutation [(2), (3), (5)] : $\begin{pmatrix}123\\ jkn\end{pmatrix}$

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